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In this report, selection theory is used as a theoretical framework from which mathematical algorithms for tailored testing are derived. The process of tailored, or adaptive, testing is presented as analogous to personnel selection and rejection on a series of continuous variables that are related to ability. Proceeding from a single common-factor model, the author derives the two- and three-parameter normal ogive item response functions as submodels. For both of these submodels, algorithms are developed for sequential item selection, ability estimation, and test termination in the

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context of adaptive ability testing. It is shown that the adaptive testing method based on these algorithms is formally identical to a previously developed Bayesian sequential tailored testing procedure.

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TAILORED TESTING THEORY AND PRACTICE: A BASIC MODEL, NORMAL OGIVE SUBMODELS, AND TAILORED TESTING ALGORITHMS

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TAILORED TESTING THEORY AND PRACTICE: A BASIC MODEL, NORMAL OGIVE SUBMODELS, AND TAILORED TESTING ALGORITHMS

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FOREWORD

A joint-service coordinated effort is underway to develop a computerized adaptive testing (CAT) system and to evaluate its potential for use in the military entrance processing stations as a replacement for the Armed Services Vocational Aptitude Battery (ASVAB) printed tests. The Navy Personnel Research and Development Center has been designated lead laboratory for this effort.

This report, which was sponsored by the Commandant of the Marine Corps (MPI-20) and performed by the Office of Personnel Managment (OPM), is the fifth in a series being issued under the CAT project. It describes a theoretical foundation on which the adaptive administration of aptitude tests may be based, and is intended for use by professionals in the field of psychometrics. The previous reports issued are listed below:

1. McBride, J. R. Computerized adaptive testing project: Objectives and requirements (NPRDC Tech. Note 82-22), July 1982.

2. Croll, P. R. Computerized adaptive testing system design: Preliminary design characteristics (NPRDC Tech. Rep. 82-52), July 1982. (AD-A118 495)

3. Wetzel, C. D., & McBride, J. R. Influence of fallible item parameters on test information during adaptive testing (NPRDC Tech. Rep. 83-15), April 1983.

4. Moreno, K. E., Wetzel, C. D., & McBride, J. R. Relationship between corresponding Armed Services Vocational Aptitude Battery (ASVAB) and computerized adaptive testing (CAT) subtests (NPRDC Tech. Rep. 83-27), August 1983.

This report was initially conceived and planned at OPM when tailored testing was proposed for federal civil service examining on a widespread basis. Dr. Neil J. Dorans played a significant part in the original planning, outlining, and writing of early versions of the report. Several persons provided valuable reviews and comments on earlier versions of the report, including Professor Hubert E. Brogden of Purdue University, Dr. Charles H. Anderson of OPM, and Professor Bert F. Green of the Johns Hopkins University. Editorial assistance was capably provided by Cynthia L. Clark of OPM, Washington, DC.

The contracting officer's technical representative was Dr. James R. McBride.

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Commanding Officer

JAMES W. TWEEDDALE
Technical Director

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INTRODUCTION

Tailored testing is a mode of pyschological testing where examinees interact with a computer to answer a series of questions, each prompted by the correctness or incorrectness of responses to previous questions. This mode of testing has become the focus of considerable activity during the past decade. The impetus for this activity has been the recognition of the potential of tailored testing to provide substantial benefits along with reductions in the overall cost of testing. Marked improvements in measurement, ease of administration, security of test materials, and examinee convenience are frequently mentioned as major potential benefits. Reductions in the overall costs of testing are possible because of recent and dramatic advances in computer technology.

This activity has produced an extensive literature. For familiarity with its scope, the interested reader is referred to the proceedings of four conferences devoted either partially or completely to tailored testing (Holtzman, 1970; Clark, 1976; Weiss, 1978, and Weiss, 1980).

The preferred basis for tailored testing is latent trait theory. In the summer of 1977, the *Journal of Educational Measurement* published a thematic issue on the subject. This issue might also be consulted for further background. In particular, one article (Urry, 1977) deals exclusively with tailored testing. In this article the author reports the results of an empirical study where tailored testing was conducted until the reliability of the test scores from this process matched the reliability of a particular conventional paper-and-pencil test. It was found that tailored testing required 80% fewer items than the conventional testing to achieve the matched reliability.

The particular algorithm used for the tailoring of tests in that empirical study will be derived, explained, and numerically illustrated in the present report. The algorithm is based on a three-parameter normal ogive submodel which, along with the two-parameter normal ogive submodel, will be formally derived in this report. These submodels are derived from the more familiar model for a single common factor. The three-parameter submodel is applicable to multiple-choice items. In certain circumstances, the data bases for multiple-choice items are now sufficient for the implementation of tailored testing. The data bases for free-response (short answer) items do not now enjoy this status. However, when tailored testing is implemented with multiple-choice data bases, the active involvement of the computer facilitates the development of free-response data bases. This involvement introduces the required flexibility into the scoring of items. Since the two-parameter normal ogive submodel is applicable to free-response (short answer) items, it then has possible future applications. Also, a discussion of the two-parameter submodel provides the necessary prologue to an understanding of the three-parameter submodel.

In this report selection theory (Lawley, 1943) is used as the theoretical framework for the development of tailored testing algorithms. This framework possesses conceptual utility. The process of tailored testing is presented as analogous to selections and rejections on a series of continuous variables that are related to ability, the variable of primary interest. In this case, the series is purposefully chosen. The purpose is to obtain subpopulations that are homogeneous with respect to ability. The binary scores on items—*one* for a correct answer and *zero* for an incorrect answer—when placed in the perspective of the model for a single common factor, ability, are viewed as analogues of selection and rejection on continuous variables that are usually observed under this basic model.

Given this theoretical as well as conceptual framework, the estimates of ability become the means of ability for the subpopulations that would have resulted from the purposeful series of selections and rejections indicated by the patterns of binary scores. The variances of these ability estimates become the restricted variances of ability for the resulting subpopulations. Precision in tailored testing is viewed as the analogue of severe restriction in range or homogeneity with respect to the variable of primary interest for the resultant subpopulations.

The exposition in this report deals with the basic model; the normal ogive submodels that derive from this basic model; the tailored testing process; the tailoring algorithms that are derived from selection theory; and the basis for the control of these algorithms in an interactive tailoring system. This basis of control also derives from selection theory.

In Chapter 1, the model for a single common factor will be reviewed. This basic model assumes that observed random variables can be decomposed into a common factor (true score) component, and a unique (error score) component. In Chapters 2 and 3 it will be assumed that normally distributed random variables, which might be observed, completely determine the binary scores on items, that is, a *zero* for an incorrect answer and a *one* for a correct answer. Given the model for a single common factor, the manner of this determination gives rise to the normal ogive submodels, in particular, the two- and three-parameter cases. Some further consequences of binary scoring will be developed for each case. These developments incorporate brief reviews of univariate and multivariate normal distributions. Some important results are then obtained which are required in subsequent derivation.

In Chapter 4 a brief introduction is provided for the tailoring process. In Chapter 5 the analogy is drawn between the assumed basis of binary scores and the process of selection or rejection. Tailored testing algorithms are then derived from selection theory for both normal ogive submodels. Each algorithm sequentially provides (a) the choice of item, (b) the ability estimate, and (c) the variance of the ability estimate. The principle of invariance is illustrated with respect to these algorithms. In Chapter 6, the recursive nature of the derived formulation is demonstrated; and the identity of this derived formulation to previously developed Bayesian procedures (Owen, 1975) is established. It is hoped that these non-Bayesian developments will improve the accessibility of these procedures for psychometricians who are comfortable with selection theory, yet unfamiliar with Bayesian statistics and terminology. In Chapter 7 the implications of selection theory for termination rules in tailored testing are discussed; and the relationship between equiprecision in ability estimation as achieved through termination rules and tailored test reliability is developed. Termination rules are derived from a particular submodel given all of its various assumptions. One then has the means of assessing the validity of the particular submodel and all of its various assumptions. These rules forecast or predict various observable results. The results predicted by the termination rules can be compared with those actually obtained. In the aforementioned study (Urry, 1977), the obtained results were predicted very well from the termination rules. In this instance the three-parameter normal ogive submodel was, thus, found to be valid for multiple-choice items. A valid submodel ensures that termination rules are informative; and informative termination rules are essential to the development of an interactive tailored testing system. This testing is conducted individually; but normative results must be predictable in order to complete the testing of each examinee effectively. The developments in this chapter will be given further elaboration in Part II, a subsequent report. This elaboration will enable multiple ability applications of tailored testing. In Chapter 8, a numerical example is provided for an individually tailored test.

A subsequent report will treat ability estimation in data sets from conventional paper-and-pencil testing; item parameter estimation for data sets from both conventional and tailored testing; multiple ability applications of tailored testing; and some indispensable allied topics—namely, simulation research for both the unidimensional and multidimensional cases and procedures for psychometric quality control. While this report is concerned mainly with basic developments, the subsequent report will deal more directly with applied considerations.

In order to understand this report, the reader should have a thorough familiarity with scalar and matrix algebra. A working knowledge of calculus would be helpful; but the expositions, particularly in Chapters 2, 3 and 5, attempt to avoid this requirement by summarizing and illustrating the important results from calculus. The derivation of these results is presented subsequently in sections entitled "Mathematical Proofs." In Chapter 5, the results of lengthy and detailed algebraic developments will also be summarized and illustrated. These lengthy developments will likewise be presented in the section entitled "Mathematical Proofs." The "Mathematical Proofs" sections may then be omitted by the reader without loss of continuity.

1. THE BASIC MODEL

1.1 A Model for a Single Common Factor

The process of tailored testing as described here entails the use of a simple model with straightforward, perhaps strong, assumptions. The model involves a single common factor. It can be viewed as a variant of a special case of a general model for the analysis of covariance structures (Joreskog, 1978), specifically a model for ''congeneric'' items.

The application of this model to test items requires some additional consideration that will be given in Chapters 2 and 3. Derivations in the present chapter will show that the common factor model assumes observed variables to be continuous variables; whereas test items typically produce binary variables. That is to say, items are scored *zero* or *one* alternatively denoting incorrect or correct answers. In order to apply the model to binary items, it is then necessary to assume that continuous random variables underlie and completely determine the binary random variables represented by binary item scores. In Chapters 2 and 3, continuous random variables are assumed to completely determine the binary random variables produced by both the free-response (short answer) item and the more familiar multiple-choice item. The manner in which a continuous random variable completely determines a binary random variable for a free-response item leads to the two-parameter normal ogive submodel. This submodel will be derived and discussed in Chapter 2. The manner in which a continuous random variable completely determines a binary random variable for a multiple-choice item leads to the three-parameter normal ogive submodel. This submodel will be derived and discussed in Chapter 3. These following chapters will provide examples of continuous random variables for free-response and multiple-choice items where the normal ogive submodels are given separate treatment.

In this report, standard notation will be observed. Random variables, whether continuous or discrete, and random vectors will be denoted by capital letters. Lowercase letters will denote realizations or scores on the random variables and random vectors.

Let Z_g be designated as the continuous random variable basic to the binary scoring of item g. Each realization of Z_g, ζ_g, is a score that is assigned to a particular answer to item g.

The location of a particular individual i on the score continuum representing the different values of the random variable for item g on a particular occasion o is symbolized by ζ_{gio}. In the classical test theory formulation of Lord and Novick (1968), ζ_{gio} is decomposed into a true score component, τ_{gi}, and an error score component, δ_{gio},

$$\zeta_{gio} = \tau_{gi} + \delta_{gio}. \tag{1.1.1}$$

Following Lord and Novick, the true score component of person i's score, τ_{gi}, is defined as the expected value of ζ_{gio} over repeated administrations of item g to individual i under identical testing conditions

$$\tau_{gi} = \mathscr{E}_o \zeta_{gio}. \tag{1.1.2}$$

The measurement error on occasion o is the simple difference,

$$\delta_{gio} = \zeta_{gio} - \tau_{gi}. \tag{1.1.3}$$

which is in mean deviate form because of (1.1.2). Some statistically desirable consequences of these particular true score and error score definitions are proved in Lord and Novick (1968, pp. 29–50).

The fundamental assumptions that characterize the model are (1) that the true score components of each continuous variable determining the binary scores on item g are linear transformations of a common underlying latent trait or factor, (2) that scores on this latent trait are normally distributed in the population of potential examinees, and (3) that measurement error is also normally distributed in the population of potential examinees. For item g, these assumptions are expressed formally via

$$T_g = \beta_g \theta, \tag{1.1.4}$$

$$\theta \sim N[\mu(\Theta), \sigma^2(\Theta)], \tag{1.1.5}$$

and

$$\Delta_g \sim N[\mu(\Delta g), \sigma^2(\Delta_g)], \tag{1.1.6}$$

where T_g and Δ_g are the true score and error variables for the continuous variable basic to item g, Θ is the latent trait variable with mean $\mu(\Theta)$, variance $\sigma^2(\Theta)$, and β_g is the weight that transforms the latent trait metric into the true score metric for the continuous variable underlying item g. Equation (1.1.4) asserts that true score on the continuous variable determining the binary scores on item g is the product of a weight β_g and the score on the latent trait Θ; while equation (1.1.5) indicates that the scores on the latent trait Θ are distributed normally with a mean of $\mu(\Theta)$ and a variance of $\sigma^2(\Theta)$. Asserted in (1.1.6) is that the error variable associated with the continous variable which determines the binary scores on item g is distributed normally with a mean of $\mu(\Delta_g)$ and a variance of $\sigma^2(\Delta_g)$. A set of equations similar to (1.1.4) and (1.1.6) exists for each of the remaining $p - 1$ items. All p sets of equations can be expressed compactly in matrix notation as

$$\boldsymbol{T} = \boldsymbol{\beta}\theta \tag{1.1.7}$$

and

$$\boldsymbol{\Delta} \sim N[\boldsymbol{\mu}(\Delta), \Sigma(\Delta,\Delta)] \tag{1.1.8}$$

where $\boldsymbol{T}$ is a p-by-1 true score vector, $\boldsymbol{\Delta}$ is a p-by-1 measurement error score vector, $\boldsymbol{\beta}$ is a p-by-1 vector of transformation weights, $\boldsymbol{\mu}(\Delta)$ is a p-by-1 vector of measurement error means, and $\Sigma(\Delta,\Delta)$ is a p-by-p matrix of measurement error variances and covariances. Equation (1.1.7) indicates that the vector of true scores on the continuous variables determining the binary item scores is the product of the vector of weights, $\boldsymbol{\beta}$, for these continuous variables and the scores on the latent trait Θ; while (1.1.8) asserts that the error score vector $\boldsymbol{\Delta}$ is distributed multinormally with a mean vector $\boldsymbol{\mu}(\Delta)$, and a variance-covariance matrix $\Sigma(\Delta,\Delta)$.

The definitions of true score and error score in (1.1.1) and (1.1.2) and the normality and linearity assumptions in (1.1.5), (1.1.7), and (1.1.8) yield the following results:

$$\boldsymbol{\mu}(\boldsymbol{\Delta}) = \mathbf{0} \text{ (mean error score on every item is zero);} \tag{1.1.9}$$

$$\rho(\Theta,\Delta_g) = 0 \text{ for all } g, \text{ (error scores and latent trait scores are uncorrelated); and} \tag{1.1.10}$$

$$\rho(\Delta_g,\Delta_h) = 0 \text{ for all } g \text{ and } h \text{ where } g \neq h, \text{ (error scores on different items are uncorrelated).} \tag{1.1.11}$$

As a consequence of (1.1.11), $\Sigma(\Delta,\Delta)$ is a diagonal matrix. The linear independencies evident in (1.1.10) and (1.1.11) imply stronger statistical independencies under the stated normality assumptions.

Since linear combinations of normally distributed variables are also normally distributed (Anderson, 1958), it follows from

$$\boldsymbol{Z} = \boldsymbol{\beta}\,\theta + \boldsymbol{\Delta} \tag{1.1.12}$$

that

$$\boldsymbol{Z} \sim N[\boldsymbol{\beta}\mu(\Theta), \Sigma(Z,Z)]. \tag{1.1.13}$$

or, in words, that the p-by-1 vector $\boldsymbol{Z}$ is distributed multinormally with a mean vector provided by the vector and scalar product $\boldsymbol{\beta}\mu(\Theta)$ and with a variance-covariance matrix $\boldsymbol{\Sigma}(Z,Z)$. This matrix contains the variances and covariances of the continuous variables Z_g that determine the binary item scores.

The p-by-p variance-covariance matrix $\Sigma(Z,Z)$ can be decomposed into

$$\Sigma(Z,Z) = \boldsymbol{\beta\beta}' \sigma^2(\Theta) + \Sigma(\Delta,\Delta). \tag{1.1.14}$$

Equations (1.1.12) and (1.1.14) can be immediately recognized as the fundamental equations of a linear common factor model with a single common factor, Θ. Since $\Sigma(\Delta,\Delta)$ is a diagonal matrix, the off-diagonal element in the gth row and hth column of $\Sigma(Z,Z)$, $\sigma(Z_g,Z_h)$, is

$$\sigma(Z_g,Z_h) = \beta_g\beta_h\sigma^2(\Theta). \tag{1.1.15}$$

Note that (1.1.12) and (1.1.14) are also standard equations from simple linear regression where the Z_g are the criteria, Θ is the predictor, and the Δ_g are the errors of prediction. As such $\boldsymbol{\beta}$ is a p-by-1 vector of least squares regression weights,

$$\boldsymbol{\beta} = [\mathrm{Diag}\ \Sigma(Z,Z)]^{.5}\boldsymbol{\rho}(Z_g,\Theta)[\sigma(\Theta)]^{-1} \tag{1.1.16}$$

where the gth diagonal element of the p-by-p diagonal matrix, $[\mathrm{Diag}\ \Sigma(Z,Z)]$, is the variance of Z_g, $\sigma^2(Z_g)$, and $\boldsymbol{\rho}(Z_g,\Theta)$ is a p-by-1 vector of correlations between the Z_g and Θ. In addition, the gth diagonal element of $\Sigma(\Delta,\Delta)$ can be expressed as

$$\sigma^2(\Delta_g) = \sigma^2(Z_g)\ [1 - \rho^2(Z_g,\Theta)]. \tag{1.1.17}$$

If the Z_g and Θ are scaled to a mean of zero and a standard deviation of unity, (1.1.16) and (1.1.17) reduce to the forms

$$\boldsymbol{\beta} = \boldsymbol{\rho}(Z_g,\Theta) \tag{1.1.18}$$

and

$$\sigma^2(\Delta_g) = 1 - \rho^2(Z_g,\Theta). \tag{1.1.19}$$

Note that the measurement error variance in (1.1.19) is also the conditional variance of Z_g given θ,

$$\sigma^2(\Delta_g) = \sigma^2(Z_g \mid \theta), \tag{1.1.20}$$

and that this conditional variance is the same at all levels of the continuous variable Θ, that is the errors of measurement are homoscedastic as a consequence of the normality assumptions. In addition, substitutions from (1.1.18) into (1.1.15) yield

$$\sigma(Z_g,Z_h) = \rho(Z_g,Z_h) = \rho(Z_g,\Theta)\ \rho(Z_h,\Theta) \tag{1.1.21}$$

because of the unit scalings of the continuous variables Z_g and Θ. Equations (1.1.18) through (1.1.21) can be recognized as aspects of the traditional correlational formulation of the model for a single common factor, and can also be viewed as characteristic of a model for "congeneric" items.

Notice that specific factors for items are not defined in this model. As a result, item-specific factors would be included in the measurement error.

2. EFFECTS OF BINARY SCORING: A SUBMODEL FOR FREE-RESPONSE ITEMS

2.1 The Two-Parameter Normal Ogive Submodel

Suppose, as did Samejima (1969), that the answers to a free-response (short answer) item can be placed in order based on the degree of attainment of a proper solution to the problem posed by the item. For p test items designed to measure the same ability, such an ordering would generate p item score continua. When the scores along these continua are associated with the relative frequencies of individuals producing each particular answer, continuous random variables such as the Z_g of Chapter 1 would be the result. Now suppose that a decision is made to separate the ordered answers on each of the p item continua into two mutually exclusive sets, acceptable (correct) and unacceptable (incorrect). This separation entails the dichotomization of the p continua. Let γ_g symbolize the point of dichotomization along the continuum Z_g. Dichotomization of Z_g is represented formally by a rescaling of the continuous variable Z_g into a binary variable U_g. By convention, correct answers are scored *one*, while incorrect answers are scored *zero*. Hence U_g has two possible realizations:

$u_g = 1$ When $\zeta_g \geqslant \gamma_g$ (correct answer produced)
$u_g = 0$ When $\zeta_g < \gamma_g$ (incorrect answer produced)

Note that the dichotomization of Z_g completely determines the random variable U_g. This dichotomization also produces two additional random variables. Since the Z_g are affected by measurement error, so are the U_g. The transformation of Z_g into U_g thus results in a transformation of true score metric. In effect, a new true score variable T_g and a new error score variable E_g are produced implicitly by the Z_g into U_g transformation.

For individual i on testing occasion o, t_{gi} represents the individual's true score on binary item g, while e_{gio} is the individual's error score on binary item g. Again, true score is defined as the expected value of the u_{gio}, which are realizations of U_g for individual i, over testing occasions o, that is,

$$t_{gi} = \mathscr{E}_o \, U_{gio} \tag{2.1.1}$$

while error score is defined as

$$e_{gio} = u_{gio} - t_{gi}. \tag{2.1.1}$$

Under the definition of true score presented in (2.1.1), T_g is the least squares estimate of U_g given θ. All individuals of ability equal to that of individual i, $\theta = \theta_i$, have identical true scores, t_g, that is

$$t_g = \mathscr{E}\,(U_g \mid \theta = \theta_i) \tag{2.1.3}$$

which is a specific realization of the random variable

$$T_g = \mathscr{E}\,(U_g \mid \theta). \tag{2.1.4}$$

The expectation relationship in (2.1.4) defines T_g as the least squares regression function of U_g onto θ.

At this point, a functional form for the regression of U_g onto θ must be determined. Due to the normality assumptions stated in Chapter 1, this functional form is the item characteristic curve for the two-parameter normal ogive of latent trait theory (Lord & Novick, 1968). The derivation of this form requires some elementary calculus and knowledge of statistical expectations.

By definition, the expected value of a discrete random variable, such as U_g, is the sum of the products between each realization of U_g and its relative frequency. The expected value of U_g given θ is,

$$\begin{aligned} T_g &= \mathscr{E}(U_g \mid \theta) = \sum_{u_{gk}=0}^{1} u_{gk} \Pr(u_{gk} \mid \theta) = (0) \Pr(u_g = 0 \mid \theta) + (1) \Pr(u_g = 1 \mid \theta) \\ &= \Pr(u_g = 1 \mid \theta) \end{aligned} \tag{2.1.5}$$

where $\Pr(u_g = 0 \mid \theta)$ and $\Pr(u_g = 1 \mid \theta)$ are the probability of an incorrect answer on item g given θ and the probability of a correct answer on item g given θ respectively. Thus, the item true score random variable, which is the least squares estimate of U_g given θ, is equivalent to the probability of a correct answer on item g given θ. Obtaining an expression for $\Pr(u_g = 1 \mid \theta)$ provides us with the functional form for the regression of U_g onto θ.

Following the notation used by Lord and Novick (1968), $\Pr(u_g = 1 \mid \theta)$ is simply the item characteristic curve for item g, $P_g(\theta)$

$$\Pr(u_g = 1 \mid \theta) = P_g(\theta). \tag{2.1.6}$$

From the definition of U_g as the result of a binary rescaling of Z_g, it is known that

$$P_g(\theta) = \Pr(u_g = 1 \mid \theta) = \Pr(Z_g \geqslant \gamma_g \mid \theta). \tag{2.1.7}$$

Therefore, the proportion of the conditional distribution of Z_g that is above the point of dichotomization γ_g equals the true score for U_g given θ. The regression function T_g can be viewed as the curve that connects these various conditional proportions that can be computed at each value of the continuous variable Θ.

Due to the normality assumptions stated in Chapter 1, the conditional distributions of Z_g are normal. Thus, the cumulative normal distribution function $\Phi[*]$ can be used to obtain the probability of a correct answer to item g given θ. To use the cumulative normal distribution function, scores on the conditional distribution of Z_g given θ must be converted to a mean of zero and a variance of unity; and γ_g, the point of dichotomization on Z_g, must be rendered in the scale of the standardized conditional distribution. Let $\tilde{\zeta}_g(\theta)$ represent scores that are standardized to a mean of zero and a variance of unity; and let $\gamma_g(\theta)$ represent the point of dichotomization on the $\tilde{Z}_g(\theta)$.

For a given value of Θ, the mean of Z_g is

$$\mu(Z_g \mid \theta) = \rho(Z_g, \Theta)\theta \tag{2.1.8}$$

and the variance of Z_g is

$$\sigma^2(Z_g \mid \theta) = 1 - \rho^2(Z_g, \Theta). \tag{2.1.9}$$

These relationships flow from the normality assumptions of Chapter 1 and the fact that the unconditional distributions of both Θ and Z_g have been scaled to a mean of zero and a variance of unity. Equations (2.1.8) and (2.1.9) enable us to define $\tilde{\zeta}_g(\theta)$ as

$$\tilde{\zeta}_g(\theta) = \frac{\zeta_g - \mu(Z_g \mid \theta)}{\sigma(Z_g \mid \theta)} = \frac{\zeta_g - \rho(Z_g, \Theta)\theta}{[1 - \rho^2(Z_g, \Theta)]^{.5}}. \tag{2.1.10}$$

The relationships in (2.1.8) and (2.1.9) also enable us to define $\gamma_g(\theta)$ as

$$\gamma_g(\theta) = \frac{\gamma_g - \mu(Z_g \mid \theta)}{\sigma(Z_g \mid \theta)} = \frac{\gamma_g - \rho(Z_g, \Theta)\theta}{[1 - \rho^2(Z_g, \Theta)]^{.5}} \tag{2.1.11}$$

where simplification of the expression in (2.1.11) is possible by introducing and defining the item parameters a_g, item discriminatory power, and b_g, item difficulty, as

$$a_g = \frac{\rho(Z_g,\Theta)}{[1 - \rho^2(Z_g,\Theta)]^{.5}} \tag{2.1.12}$$

and

$$b_g = \frac{\gamma_g}{\rho(Z_g,\Theta)} . \tag{2.1.13}$$

Given (2.1.13), an explicit solution for γ_g is provided by

$$\gamma_g = \rho(Z_g,\Theta)b_g. \tag{2.1.14}$$

Substitutions from (2.1.14) and then (2.1.12) into the rightmost equality in (2.1.11) lead to

$$\gamma_g(\theta) = - a_g(\theta - b_g) \tag{2.1.15}$$

as the point of dichotomization on $\tilde{Z}_g(\theta)$.

In terms of $\tilde{Z}_g(\theta)$ and $\gamma_g(\theta)$, the probability of a correct answer to item g given θ can be expressed as

$$\begin{aligned} P_g(\theta) &= \Pr[\tilde{Z}_g(\theta) \geqslant \gamma_g(\theta)] \\ &= \int_{\gamma_g(\theta)}^{\infty} (2\pi)^{-.5} \exp\{-.5[\tilde{\zeta}_g(\theta)]^2\}\, d\tilde{\zeta}_g(\theta) \end{aligned} \tag{2.1.16}$$

or, equivalently, due to symmetry of the normal distribution function,

$$P_g(\theta) = \int_{-\infty}^{-\gamma_g(\theta)} (2\pi)^{-.5} \exp\{-.5[\tilde{\zeta}_g(\theta)]^2\}\, d\tilde{\zeta}_g(\theta). \tag{2.1.17}$$

Given (2.1.15), it is to be noted that

$$P_g(\theta) = \Phi[- \gamma_g(\theta)] = \Phi[a_g(\theta - b_g)] \tag{2.1.18}$$

where $\Phi[*]$ is by definition the cumulative normal distribution function as conventionally expressed in (2.1.17) as the area in the standard normal distribution between negative infinity and, in this case, negative $\gamma_g(\theta)$.

The Parameters

In (2.1.18), one has an expression for the item characteristic curve, the item true score function T_g, and the regression of the free-response, binary item U_g onto latent ability Θ. This functional form gives the probability of a correct answer on item g as a function of the continuous variable Θ. The parameters of this expression are a_g and b_g, which are defined in (2.1.12) and (2.1.13).

A fuller appreciation of these item parameters is acquired through examination of Figures 2.1.1 and 2.1.2. Latent ability, Θ, is portrayed along the abscissa in both figures. Portrayed along the ordinate in Figure 2.1.1 is the dichotomized continuous variable Z_g. The point of dichotomization on Z_g, γ_g, is located on the ordinate. The line segment extending from γ_g across the figure, parallel to the abscissa, divides each conditional distribution of $\tilde{Z}_g(\theta)$ at γ_g into two mutually exclusive areas, $P_g(\theta)$ and $Q_g(\theta)$, which represent the probabilities of correct and incorrect answers to item g, given θ, where the point of dichotomization, γ_g, has been rendered in the scale of $\tilde{Z}_g(\theta)$ as $\gamma_g(\theta)$.

The line traversing the figure at an angle of less than 45 degrees with the abscissa represents the linear regression of Z_g onto Θ. When, as in this case, both Θ and Z_g are scaled to a mean of zero and a variance of unity, the slope of this linear regression is $\rho(Z_g, \Theta)$, the correlation between Z_g and Θ.

The item parameter a_g is defined in (2.1.12) as the ratio of the slope of the regression of Z_g onto Θ to the standard deviation of the conditional distributions of Z_g, which is the same at all levels of Θ due to the normality assumptions of Chapter 1. It may be observed that the squaring of a_g yields a signal-to-noise ratio for the linear regression of Z_g onto Θ,

$$a_g^2 = \frac{\rho^2(Z_g, \Theta)}{1 - \rho^2(Z_g, \Theta)} . \tag{2.1.19}$$

Notice in Figure 2.1.1 where the horizontal line extending from γ_g intersects the line of regression of slope $\rho(Z_g, \Theta)$. The projection of this point of intersection onto the latent trait continuum defines the second parameter, b_g. Since the conditional distributions of Z_g are symmetrical around the line of regression, the total area of the conditional probability distribution of Z_g at $\theta = b_g$ is halved by the line extending from γ_g, across the figure. Hence, at $\theta = b_g$, $P_g(\theta) = Q_g(\theta) = .5$ where $Q_g(\theta)$, the complement of $P_g(\theta)$, is defined by

$$\begin{aligned} Q_g(\theta) = 1 - P_g(\theta) &= \int_{-\infty}^{\gamma_g(\theta)} (2\pi)^{-.5} \exp\{-.5[\tilde{\zeta}_g(\theta)]^2\} \, d\tilde{\zeta}_g(\theta) \\ &= \Phi[\gamma_g(\theta)] = \Phi[-a_g(\theta - b_g)] \end{aligned} \tag{2.1.20}$$

which yields the probability of an incorrect answer to item g. Again $\Phi[*]$ is by definition the cumulative normal distribution function, or the area under the standard normal curve between negative infinity and, in this case, $\gamma_g(\theta)$.

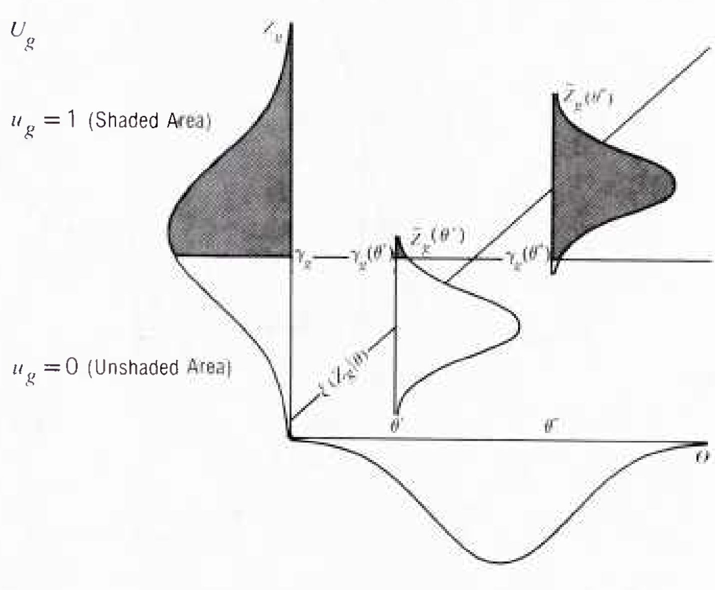

Figure 2.1.1. Hypothetical relations among the item continuum Z_g, the free-response, binary item U_g, and the latent trait Θ.

Figure 2.1.2 represents the other regression of interest in this submodel, the nonlinear regression of the binary item U_g onto the continuous latent trait, Θ. From (2.1.5) through (2.1.7) it is known that the regression of U_g onto Θ is represented by $P_g(\theta)$, the probability of a correct answer to item g given θ. Hence the curve of regression in Figure 2.1.2 may be obtained by computing the areas of the conditional distributions of Z_g given θ at levels of Θ, such as those depicted by the shaded portions in Figure 2.1.1, and then plotting this infinite number of areas against the abscissa to obtain Figure 2.1.2. In practice integral calculus is used to make the transition from Figure 2.1.1 to Figure 2.1.2.

Example 2.1.1. Item g as depicted in Figure 2.1.1 can be used for the purpose of numerical illustration. The slope of the regression of Z_g onto Θ, $\mathscr{E}(Z_g \mid \theta)$, is .90. This value is also $\rho(Z_g,\Theta)$ because both Z_g and Θ are depicted as distributed with a mean of zero and a variance of unity. Thus item discriminatory power, a_g, as given by (2.1.12) is 2.0647. In Figure 2.1.1 the ordinate and abscissa are both portrayed for the range of values from -3.0 to $+3.0$. The projection of the point of intersection of the line segment extending horizontally through γ_g and the line of regression, $\mathscr{E}(Z_g \mid \theta)$, onto the abscissa, occurs at a value of $-.375$. Thus b_g is equal to $-.375$. Remember the conditional distributions of Z_g given θ are symmetric about their means, which coincide with the line of regression, $\mathscr{E}(Z_g \mid \theta)$; and by definition, at $\theta = b_g$, $P_g(b_g)$, the probability of a correct answer to item g is .5, as indicated in Figure 2.1.2.

Example 2.1.2. In Figures 2.1.1 and 2.1.2, θ' has the value of -1.50. When $\gamma_g(\theta)$ of (2.1.15) is evaluated at θ', it is found that $\gamma_g(\theta')$ equals 2.3228. The area above $\gamma_g(\theta')$ on $\tilde{Z}_g(\theta')$, the standardized conditional distribution of Z_g given θ', is provided by (2.1.18) as .0101. This value of $P_g(\theta')$ is represented by the size of the shaded area in the standardized conditional distribution $\tilde{Z}_g(\theta')$ as depicted in Figure 2.1.1. This value is the probability of a correct answer to item g at a standard ability score of θ' or -1.50. Hence, a point has been placed in Figure 2.1.2 with the coordinates of .0101 on the ordinate, $P_g(\theta)$, and θ', -1.50, on the abscissa.

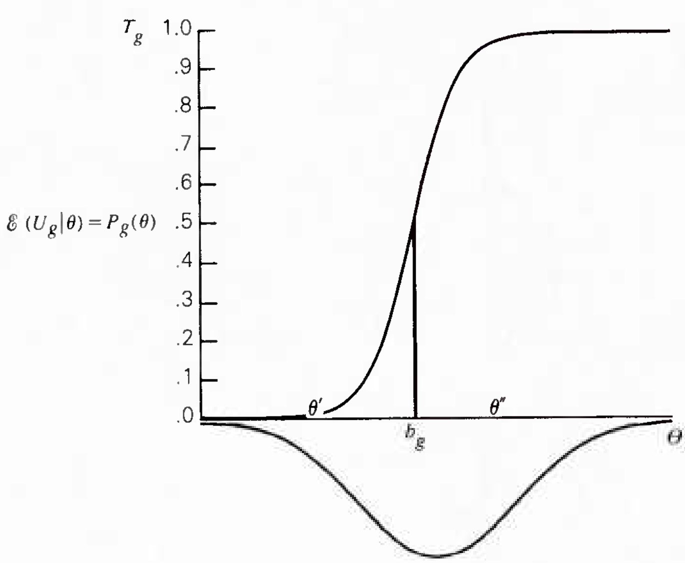

Figure 2.1.2. The item characteristic curve or the regression of a free-response, binary item U_g on the latent trait Θ.

Example 2.1.3. In Figures 2.1.1 and 2.1.2, θ'' has the value of $+.75$. When $\gamma_g(\theta)$ of (2.1.15) is evaluated at θ'', it is found that $\gamma_g(\theta'')$ equals -2.3228. The area above $\gamma_g(\theta'')$ on $\tilde{Z}_g(\theta'')$, the standardized conditional distribution of Z_g given θ'', is given by (2.1.18) as .9899. This value of $P_g(\theta'')$ is represented by the size of the shaded area in the standardized conditional distribution $\tilde{Z}_g(\theta'')$ as portrayed in Figure 2.1.1. This value represents the probability of a correct answer to item g at a standardized score of θ''or $+.75$. Again, a point has been placed in Figure 2.1.2 with the coordinates of .9899 on the ordinate, $P_g(\theta)$, and θ'', $+.75$, on the abscissa.

The item characteristic curve of Figure 2.1.2 can be constructed by repeating the numerical process illustrated in Examples 2.1.2 and 2.1.3 for a sufficient number of points equally spaced on the abscissa. It is often helpful to be able to inspect the item characteristic curves for various items given their parameters. These item characteristic curves can be conveniently produced for this purpose through the use of a plotter and the formulation as illustrated for the arbitrary points θ' and θ'' for a free-response item g.

Examination of Figure 2.1.2 provides us with additional information about the latent trait item parameters a_g and b_g. In mathematical terms, b_g is the value on the latent ability continuum corresponding to the point of inflection on the item characteristic curve, and a_g is proportional to the slope of the item characteristic curve at its point of inflection. As a value expressed on the Θ continuum, b_g is often referred to as the item location parameter. Note that as the location of the curve shifts to the right along the Θ continuum, b_g increases, requiring greater ability to maintain the same probability of successful performance on the item. Hence, b_g is most commonly known as the item difficulty parameter.

The parameter a_g indicates how well the item discriminates between levels of ability that are slightly above and below $\theta = b_g$. As a_g gets larger, the slope of the curve becomes steeper and the discrimination between ability levels close to b_g increases. In contrast, flatter curves have lower values of a_g, reflecting coarser measurement over a broader range of ability. Thus a_g is most commonly known as the parameter of item discriminatory power. For other discussion of these two parameters, Lord and Novick (1968) and Hambleton and Cook (1977) can be consulted.

2.2 Further Consequences of the Submodel

Given the basic model from which the two-parameter normal ogive submodel is derived, it is known that the joint distribution of Z_g and Θ is bivariate normal. This condition of bivariate normality allows the derivation of various convenient mathematical expressions. The derivations for these expressions follow from defining relationships within the bivariate normal distribution. In this section, these expressions will be presented and numerically illustrated, using free-response item g as portrayed in Figure 2.1.1 for illustrative purposes. These expressions will include the mathematical formulation for:

1. The unconditional probabilities for the realizations of U_g.
   (a) The probability of a correct answer to item g, $\Pr(u_g = 1)$.
   (b) The probability of an incorrect answer to item g, $\Pr(u_g = 0)$.
2. The conditional means of Z_g given the realizations of U_g.
   (a) The mean of Z_g given a correct answer to item g, $\mu(Z_g \mid u_g = 1)$.
   (b) The mean of Z_g given an incorrect answer to item g, $\mu(Z_g \mid u_g = 0)$.
3. The conditional variances of Z_g given the realizations of U_g.
   (a) The variance of Z_g given a correct answer to item g, $\sigma^2(Z_g \mid u_g = 1)$.
   (b) The variance of Z_g given an incorrect answer to item g, $\sigma^2(Z_g \mid u_g = 0)$.
4. The least squares estimators of ability given the realizations of U_g.
   (a) Specific case.
      (1) The mean of Θ given a correct answer to item g, $\mu(\Theta \mid u_g = 1)$.
      (2) The mean of Θ given an incorrect answer to item g, $\mu(\Theta \mid u_g = 0)$.
   (b) General case.
      (1) The mean of Θ^* given a correct answer to item g, $\mu(\Theta^* \mid u_g = 1)$.
      (2) The mean of Θ^* given an incorrect answer to item g, $\mu(\Theta^* \mid u_g = 0)$.

These expressions will be derived in Section 2.3. For those readers seeking a general understanding, Section 2.3 may be omitted without loss of continuity.

The Unconditional Probabilities for the Realizations of U_g

The binary random variable U_g can realize one of two possible values. In the case of a correct answer to free-response item g, u_g, the realization of the binary random variable U_g, equals *one*; or the realization of U_g, u_g, equals *zero* in the case of an incorrect answer to item g.

The Probability of a Correct Answer to Item g, $\Pr(u_g = 1)$. This unconditional probability, most commonly known as the p-value for item g, is designated as P_g. A convenient expression for this probability is

$$\Pr(u_g = 1) = P_g = \Phi[-\gamma_g] \tag{2.2.1}$$

where $\Phi[-\gamma_g]$ is, by definition, the cumulative normal distribution function evaluated for the argument which is the negative of γ_g, the point of dichotomization on the continuous variable Z_g. This point of dichotomization was defined in (2.1.14) as

$$\gamma_g = \rho(Z_g,\Theta)\, b_g. \tag{2.2.2}$$

In (2.2.2), $\rho(Z_g,\Theta)$ is the correlation between the continuous variables Z_g and Θ, and b_g is the item difficulty parameter.

> *Example 2.2.1.* From Example 2.1.1, it is known that $\rho(Z_g,\Theta)$ is .90 and that b_g is $-.375$ for item g as depicted in Figure 2.1.1. It is found through the use of (2.2.2) that γ_g equals $-.3375$. When P_g of (2.2.1) is evaluated for γ_g equal to $-.3375$, it is known that $\Phi[-\gamma_g]$, the cumulative normal distribution function given the argument $-\gamma_g$, equals .6321. This value of P_g represents the size of the shaded area in the marginal distribution of Z_g as portrayed along the ordinate in Figure 2.1.1. Notice that this value .6321 is the probability that ζ_g is equal to or greater than γ_g, or, synonomously, the probability of the realization that u_g equals *one*.

The Probability of an Incorrect Answer to Item g, $\Pr(u_g = 0)$. This unconditional probability is designated as Q_g. A convenient expression for this probability is

$$\Pr(u_g = 0) = Q_g = \Phi[\gamma_g] \tag{2.2.3}$$

where $\Phi[\gamma_g]$ is, by definition, the cumulative normal distribution function given the argument γ_g, and γ_g, the point of dichotomization on the continuous variable Z_g, is provided by (2.2.2).

> *Example 2.2.2.* It is known from Example 2.2.1 that γ_g is $-.3375$ for item g as presented in Figure 2.1.1. When Q_g of (2.2.3) is evaluated given γ_g equal to $-.3375$, it is found that $\Phi[\gamma_g]$, the cumulative normal distribution function given the argument γ_g, is equal to .3679. This value of Q_g represents the size of the unshaded area in the marginal distribution of Z_g as portrayed along the ordinate in Figure 2.1.1. Notice that this value .3679 is the probability that ζ_g is less than γ_g or, synonomously, the probability of the realization that u_g equals *zero*.

If γ_g were actually the cut score on an observed continuous variable Z_g, then P_g would be the selection ratio or the probability of being selected; and Q_g would be the probability of being rejected. There exists then an analogous relationship between the binary scores on item g, either a *one* or a *zero*, and selection or rejection on an observed continuous variable.

The Conditional Means of Z_g Given the Realizations of U_g

The mean of Z_g can assume only one of two possible values. In the case of a correct answer to free-response item g, u_g, the realization of the binary random variable U_g, equals *one*. Thus the mean of Z_g given a correct answer to free-response item g is designated $\mu\,(Z_g \mid u_g = 1)$. In the case of an incorrect answer to free-response item g, u_g, the realization of the binary random variable U_g, equals *zero*. Thus the mean of Z_g given an incorrect answer to free-response item g is designated as $\mu\,(Z_g \mid u_g = 0)$.

The Mean of Z_g Given a Correct Answer to Item g, $\mu(Z_g \mid u_g = 1)$. A convenient expression for this mean is given by

$$\mu(Z_g \mid u_g = 1) = \frac{\phi(\gamma_g)}{P_g}. \tag{2.2.4}$$

In (2.2.4), $\phi(\gamma_g)$ is the density of the standard normal distribution at γ_g or

$$\phi(\gamma_g) = (2\pi)^{-.5} \exp(-.5\ \gamma_g^2) \tag{2.2.5}$$

where γ_g, the point of dichotomization on the continuous variable Z_g, is provided by (2.2.2); and P_g is the unconditional probability of a correct answer to item g as given by (2.2.1).

Example 2.2.3. In Example 2.2.1, it was found that γ_g equalled $-.3375$ for item g as depicted in Figure 2.1.1. When (2.2.5) is evaluated for γ_g equal to $-.3375$, it is known that $\phi(-.3375)$, the density of the standard normal distribution evaluated at γ_g, is .3769. In Example 2.2.1, it was found that P_g equalled .6321. Evaluating (2.2.4) with these values for $\phi(\gamma_g)$ and P_g, it is known that the mean of Z_g for the subpopulation obtaining a correct answer on this item g, $\mu(Z_g \mid u_g = 1)$, is .5963. Remember that the range of the continuous variable Z_g as depicted in Figure 2.1.1 is from -3.0 to $+3.0$. This value, .5963, represents the mean of the shaded portion in the marginal distribution of Z_g. The reader can judge the accuracy of this value through visual inspection of Figure 2.1.1.

The Mean of Z_g Given an Incorrect Answer to Item g, $\mu(Z_g \mid u_g = 0)$. A convenient expression for this mean is provided by

$$\mu(Z_g \mid u_g = 0) = -\frac{\phi(\gamma_g)}{Q_g} \tag{2.2.6}$$

where $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at γ_g, is defined in (2.2.5) and Q_g, the unconditional probability of an incorrect answer to item g, is provided by (2.2.3).

Example 2.2.4. For the item presented in Figure 2.1.1, it was found in Example 2.2.3 that $\phi(\gamma_g)$ equalled .3769 and in Example 2.2.2 that Q_g equalled .3679. When (2.2.6) is evaluated for these values for $\phi(\gamma_g)$ and Q_g, it is known that the mean of Z_g for the subpopulation obtaining an incorrect answer on this item g, $\mu(Z_g \mid u_g = 0)$, is -1.0245. This value, -1.0245, represents the mean of Z_g for the unshaded portion in the marginal distribution of Z_g as portrayed in Figure 2.1.1. The reader can judge the accuracy of this value through visual inspection of Figure 2.1.1. The marginal distribution of Z_g is depicted in Figure 2.1.1 as ranging in value from -3.0 to $+3.0$.

If γ_g were actually a cut score on an observed variable Z_g, then $\mu(Z_g \mid u_g = 1)$ would be the mean of Z_g for the subpopulation explicitly selected on this variable. Similarly, $\mu(Z_g \mid u_g = 0)$ would be the mean of Z_g for the subpopulation explicitly rejected on this variable.

The Conditional Variances of Z_g Given the Realizations of U_g

The variance of Z_g can assume only one of two possible values. For a correct answer to free-response item g, the realization of the binary random variable U_g, u_g, is *one*. Hence, the variance of Z_g given a correct answer to free-response item g is designated $\sigma^2(Z_g \mid u_g = 1)$. For an incorrect answer to free-response item g, the realization of the binary random variable U_g, u_g, is *zero*. Hence, the variance of Z_g given an incorrect answer to free-response item g is designated $\sigma^2(Z_g \mid u_g = 0)$.

The Variance of Z_g Given a Correct Answer to Item g, $\sigma^2(Z_g \mid u_g = 1)$. A convenient expression for this variance is given by

$$\sigma^2(Z_g \mid u_g = 1) = 1 - \frac{\phi(\gamma_g)}{P_g}\left[\frac{\phi(\gamma_g)}{P_g} - \gamma_g\right] \tag{2.2.7}$$

where γ_g, the point of dichotomization on the continuous variable Z_g, is provided by (2.2.2); $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at γ_g, is provided by (2.2.5); and P_g, the unconditional probability of a correct answer to item g, is given by (2.2.1).

> *Example 2.2.5.* It is known from Example 2.2.1 that γ_g equals $-.3375$ and P_g equals .6321 for item g as presented in Figure 2.1.1. It is further known from Example 2.2.3 that $\phi(\gamma_g)$ equals .3769. When (2.2.7) is evaluated with these values, it is found that the variance of Z_g given a correct answer to this item g, $\sigma^2(Z_g \mid u_g = 1)$, equals .4432. The square root of this value, .6658, thus represents the standard deviation of Z_g given a correct answer to this item g, $\sigma(Z_g \mid u_g = 1)$. The value of .6658 represents the standard deviation for the shaded area of the marginal distribution of Z_g as depicted along the ordinate in Figure 2.1.1.

The Variance of Z_g Given an Incorrect Answer to Item g, $\sigma^2(Z_g \mid u_g = 0)$. A convenient expression for this variance is provided by

$$\sigma^2(Z_g \mid u_g = 0) = 1 - \frac{\phi(\gamma_g)}{Q_g}\left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g\right] \tag{2.2.8}$$

where γ_g, the point of dichotomization on the continuous variable Z_g, is provided by (2.2.2); $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at γ_g, is yielded by (2.2.5); and Q_g, the unconditional probability of an incorrect answer to item g, is given by (2.2.3).

> *Example 2.2.6.* For item g, as depicted in Figure 2.1.1, it is known from Example 2.2.1 that γ_g equals $-.3375$. It is further known from Example 2.2.2 that Q_g equals .3679 and from Example 2.2.3 that $\phi(\gamma_g)$ equals .3769. When (2.2.8) is evaluated with these values, it is found that the variance of Z_g given an incorrect answer to item g, $\sigma^2(Z_g \mid u_g = 0)$ is .2962. Thus, the square root of this value, .5443, represents the standard deviation of Z_g given an incorrect answer to this item g, $\sigma(Z_g \mid u_g = 0)$. The value .5443 represents the standard deviation of the unshaded portion of the marginal distribution of Z_g as portrayed along the ordinate in Figure 2.1.1.

The Least Squares Estimators of Ability Given the Realizations of U_g

Here two cases are considered: the specific case where ability, Θ, has a mean, $\mu(\Theta)$, of zero and a variance, $\sigma^2(\Theta)$, of unity; and the general case where ability, Θ^*, has a mean, $\mu(\Theta^*)$, and a variance, $\sigma^2(\Theta^*)$, that may be prescribed by the practitioner for convenience in applications.

Specific Case. In this case, the estimators of ability, Θ, depend on the realizations of the random binary variable U_g or the correctness or incorrectness of the answer to free-response item g. If the answer is correct, the estimator of ability is the mean of Θ for the subpopulation that would obtain a correct answer on free-response item g. If the answer is incorrect, the estimator of ability is the mean of Θ for the subpopulation that would obtain an incorrect answer on free-response item g.

Specific Case: The Mean of Θ Given a Correct Answer to Item g, $\mu(\Theta \mid u_g = 1)$. This estimator of ability is a least squares estimator because this mean provides that value of Θ about which the sum of the squared discrepancies is minimized for the variable of ability given a correct answer to free-response item g. A convenient expression for this estimator of ability is provided by

$$\mu(\Theta \mid u_g = 1) = \rho(Z_g,\Theta)\ \mu(Z_g \mid u_g = 1) = \rho(Z_g,\Theta)\,\frac{\phi(\gamma_g)}{P_g}, \tag{2.2.9}$$

where $\rho(Z_g,\Theta)$ is the correlation between the continuous variables Z_g and Θ. Notice that $\rho(Z_g,\Theta)$ is known when a_g, the item parameter of discriminatory power, is known through

$$\rho(Z_g,\Theta) = \frac{a_g}{(1 + a_g^2)^{.5}}, \tag{2.2.10}$$

which represents the explicit solution for $\rho(Z_g,\Theta)$ as derived from (2.1.12). In (2.2.9), $\mu(Z_g \mid u_g = 1)$, the mean of Z_g given a correct answer to item g, is provided by (2.2.4); γ_g, the point of dichotomization on the continuous variable Z_g, is yielded by (2.2.2); $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at the point of dichotomization, γ_g, is given by (2.2.5); and P_g, the unconditional probability of a correct answer to item g, is provided by (2.2.1).

> *Example 2.2.7.* In Example 2.1.1 it was found for item g as portrayed in Figure 2.1.1 that the parameter of discriminatory power for this item, a_g, was 2.0647. When (2.2.10) is evaluated given this value for a_g, it is found that $\rho(Z_g,\Theta)$ equals .90, which was the value obtained earlier in Example 2.1.1 through inspection of Figure 2.1.1. From Example 2.2.3, it is known that the mean of Z_g given a correct answer to item g, $\mu(Z_g \mid u_g = 1)$, is .5963. Given these values for $\rho(Z_g, \Theta)$ and $\mu(Z_g \mid u_g = 1)$, an evaluation of (2.2.9) yields .5367 as the value of the mean of Θ given a correct answer to this item g. The value .5367 represents the mean of the distribution portrayed in Figure 2.2.1.

The distribution in this figure is easily constructed. The value of the ordinate in this distribution, $f(\theta \mid u_g = 1)$, at a particular value of the abscissa, θ, given a correct answer to item g, can be obtained from

$$f(\theta \mid u_g = 1) = P_g(\theta)\ \phi(\theta) \tag{2.2.11}$$

for the range of values of Θ from negative to positive infinity. The ordinate is the product of $P_g(\theta)$, the probability of a correct answer to item g, as provided by (2.1.18), and $\phi(\theta)$, the density function for the assumed distribution of Θ when both of these terms have been evaluated at a particular value of Θ. The density function for the standard normal distribution, $\phi(\theta)$, is defined as

$$\phi(\theta) = (2\pi)^{-.5} \exp(-.5\ \theta^2). \tag{2.2.12}$$

When the values of the ordinate as obtained from (2.2.11) have been plotted with respect to the abscissa for the range of values of $\Theta(u_g = 1)$ from -3.0 to $+3.0$, the curve delineating the distribution portrayed in Figure 2.2.1 is the result.

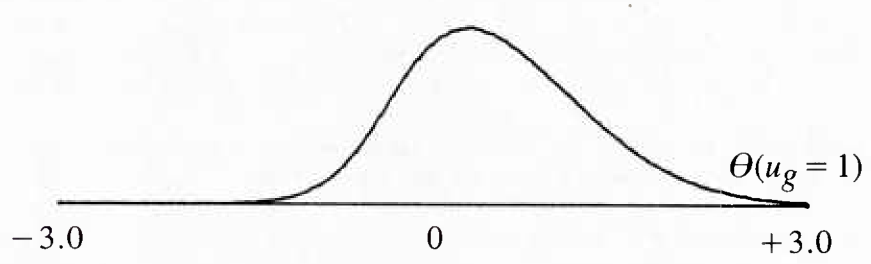

Figure 2.2.1. The distribution of ability given u_g equal to *one*, $\Theta(u_g = 1)$, or, identically, the distribution of Θ resulting from incidental selection on Θ due to explicit selection on the continuous variable Z_g where γ_g is the cut score.

Both (2.2.11) and (2.2.12) provide values of the ordinate for a particular distribution. In the instance of (2.2.11) the distribution is that of ability given a correct answer to item g, $f(\theta \mid u_g = 1)$; and, in the instance of (2.2.12), the distribution is that of ability, Θ, as originally assumed. While (2.2.12) is the density function for the distribution of Θ, (2.2.11) is not the density function for the distribution of $\Theta(u_g = 1)$. By definition, a density function provides values of the ordinate given the condition that the area under the curve is unity. The area under the curve described by (2.2.11) is known from integral calculus to be P_g, the unconditional probability of a correct answer to item g. The density function for the distribution of $\Theta(u_g = 1)$, $\phi(\theta \mid u_g = 1)$ can be obtained from (2.2.11) through

$$\phi(\theta \mid u_g = 1) = \frac{f(\theta \mid u_g = 1)}{P_g} \tag{2.2.13}$$

by dividing the outputs of (2.2.11) by the value of P_g.

It has been observed that γ_g, the point of dichotomization on the continuous variable Z_g, is translated through (2.1.11) and (2.1.15) into the $\gamma_g(\theta)$, the points of dichotomization on the standardized conditional distributions of Z_g given θ, the $Z_g(\theta)$. Remember, this translation allowed the use of the cumulative normal distribution function, $\Phi[*]$, of (2.1.18), to obtain $P_g(\theta)$, the probability of a correct answer to item g given θ. Now if γ_g were actually a cut score on an observed continuous variable Z_g, then the $\gamma_g(\theta)$ would be the corresponding cut scores on the standardized conditional distributions of Z_g given θ, the $Z_g(\theta)$. Under this interpretation, $P_g(\theta)$, as obtained from (2.1.18), yields the probability of being selected given the continuous variable Θ. As a further consequence of this interpretation, the distribution portrayed in Figure 2.2.1 as generated from (2.2.11) would be the distribution of Θ resulting from incidental selection on the continuous variable Θ due to explicit selection on the continuous variable Z_g where the cut score is γ_g.

Specific Case: The Mean of Θ Given an Incorrect Answer to Item g, $\mu(\Theta \mid u_g = 0)$. This estimator is a least squares estimator because this mean provides that value of Θ about which the sum of squared discrepancies is minimized for the variable of ability given an incorrect answer to free-response item g. A convenient expression for this least squares estimator is provided by

$$\mu(\Theta \mid u_g = 0) = \rho(Z_g,\Theta)\ \mu(Z_g \mid u_g = 0) = -\rho(Z_g,\Theta)\,\frac{\phi(\gamma_g)}{Q_g}. \tag{2.2.14}$$

In (2.2.14), $\rho(Z_g,\Theta)$ is the correlation between the continuous variables Z_g and Θ which can be obtained from (2.2.10) when a_g, the item parameter of discriminatory power, is known; $\mu(Z_g \mid u_g = 0)$ is the mean of Z_g given an incorrect answer to item g as provided by (2.2.6); γ_g is the point of dichotomization on the continuous variable Z_g as defined by (2.2.2); $\phi(\gamma_g)$ is the density in the standard normal distribution evaluated at γ_g as given by (2.2.5); and Q_g, the unconditional probability of an incorrect answer to item g is provided by (2.2.3).

> *Example 2.2.8.* In Example 2.2.7, it was found for item g as depicted in Figure 2.1.1 that $\rho(Z_g,\Theta)$, the correlation between the continuous variables Z_g and Θ, was .90; and in Example 2.2.4, the value of -1.0245 was obtained for $\mu(Z_g \mid u_g = 0)$, the mean of Z_g given an incorrect answer to item g. When these values are used in an evaluation of (2.2.14), it is found that the mean of Θ given an incorrect answer to this item g is $-.9220$. This value represents the mean of the distribution illustrated in Figure 2.2.2.

The distribution in this figure is also easily constructed. The value of the ordinate in this distribution, $f(\theta \mid u_g = 0)$, at a particular value of the abscissa, θ, given an incorrect answer to item g, can be obtained from

$$f(\theta \mid u_g = 0) = Q_g(\theta)\phi(\theta) \tag{2.2.15}$$

for the range of values of Θ from negative to positive infinity. The ordinate is the product of $Q_g(\theta)$, the probability of an incorrect answer to item g, and $\phi(\theta)$, the density function for the assumed distribution of Θ when both of these terms have been evaluated at a particular value of Θ. The probability of an incorrect answer to item g given θ, $Q_g(\theta)$, is provided by (2.1.20), and the density function for the standard normal distribution is defined by (2.2.12). When the values of the ordinate as obtained from (2.2.15) have been plotted with respect to the abscissa for the range of values of Θ from -3.0 to $+3.0$, the result is the curve delineating the distribution presented in Figure 2.2.2.

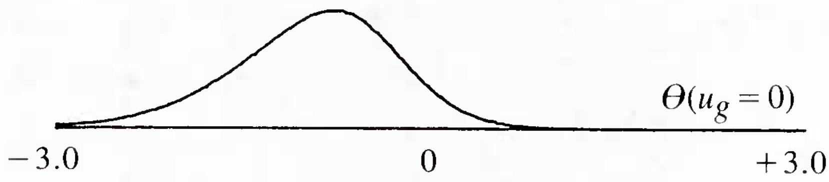

Figure 2.2.2. The distribution of ability given u_g equal to *zero*, $\Theta(u_g = 0)$, or, identically, the distribution of Θ resulting from incidental rejection on Θ due to explicit rejection on the continuous variable Z_g where γ_g is the cut score.

When divided by the appropriate value, (2.2.15), as in the instance of (2.2.12), becomes the density function for the distribution of $\Theta(u_g = 0)$ where the area under the curve is unity. As known through integral calculus, the appropriate value in this situation is Q_g, the unconditional probability of an incorrect answer to item g. This density function, $\phi(\theta \mid u_g = 0)$, is provided by

$$\phi(\theta \mid u_g = 0) = \frac{f(\theta \mid u_g = 0)}{Q_g}, \tag{2.2.16}$$

which yields the ordinate for the distribution of $\Theta(u_g = 0)$ where the area under the curve is unity.

Earlier it was noted that γ_g, the point of dichotomization on the continuous variable Z_g, is translated through (2.1.11) and (2.1.15) into the $\gamma_g(\theta)$, the points of dichotomization on the standardized conditional distributions of Z_g given θ, the $Z_g(\theta)$. Now if γ_g is again considered as the cut score on an observed continuous variable Z_g, then again the $\gamma_g(\theta)$ become the corresponding cut scores on the $\bar{Z}_g(\theta)$. Under this interpretation, $Q_g(\theta)$, as obtained from (2.1.20), yields the probability of being rejected given the continuous variable Z_g; and the distribution depicted in Figure 2.2.2 as generated from (2.2.15) would be the distribution of Θ resulting from incidental rejection on the continuous variable Θ due to explicit rejection on the continuous variable Z_g where the cut score is γ_g.

General Case. In the general case of least squares estimators of ability the practitioner prescribes the mean and variance of Θ^*, the continuous variable of ability. This prescription may be made in order to produce ability estimates yielding a given mean and variance. A certain mean and variance may be convenient in a particular application. How this prescription is made and how it is used is illustrated in Chapter 8. Under the illustrated practice, the mean of Θ^* and the mean of the estimates of Θ^* will, in expectation, be equal. However, the variance of Θ^* and the variance of the estimates of Θ^* will differ. This difference will be smaller when the ability estimates for all individuals are more precise, and larger when the ability estimates for all individuals are less precise. Thus one must hold the degree of precision fixed when solving for the variance of Θ^* that will yield the desired variance for the estimates of Θ^*. In Chapter 8, the solution for the appropriate variance of Θ^*

given the desired variance for the estimates of Θ^* is obtained by holding the degree of precision constant. In practice, this degree of precision is maintained through the effective use of termination rules as described in that chapter.

The parameters a_g, item discriminatory power, and b_g, item difficulty, were defined by (2.1.12) and (2.1.13). These parameters apply in the specific case when the continuous variable of ability, Θ, has a mean, $\mu(\Theta)$, of zero and a variance, $\sigma^2(\Theta)$, of unity. When, as in the general case, the continuous variable of ability, Θ^*, has a mean, $\mu(\Theta^*)$, and a variance, $\sigma^2(\Theta^*)$, prescribed by the practitioner, the parameters a_g of (2.1.12) and b_g of (2.1.13) no longer apply. These parameters need to be transformed to obtain the corresponding parameters a_g^* and b_g^* that are appropriate for the continuous variable Θ^*.

In this general case, a_g^*, item discriminatory power, is obtained from the transformation

$$a_g^* = \frac{a_g}{\sigma(\Theta^*)} ; \tag{2.2.17}$$

and b_g^*, item difficulty, is obtained from the transformation

$$b_g^* = b_g \, \sigma(\Theta^*) + \mu(\Theta^*), \tag{2.2.18}$$

after $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$ have been prescribed by the practitioner. These transformations yield identical probabilities of a correct answer to item g for each value of Θ and its corresponding value of Θ^*. The item characteristic curve remains invariant given the transformation of the continuous variable Θ into the continuous variable Θ^*.

Notice that the transformations in (2.2.17) and (2.2.18) are reversible. The use of

$$a_g = a_g^* \, \sigma(\Theta^*) \tag{2.2.19}$$

and

$$b_g = \frac{b_g^* - \mu(\Theta^*)}{\sigma(\Theta^*)} \tag{2.2.20}$$

will return the item parameters a_g and b_g that are appropriate for Θ when a_g^*, b_g^*, $\mu(\Theta^*)$, and $\sigma(\Theta^*)$ are known.

When a_g^* and b_g^* are known, that is, after $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$ have been prescribed, the correlation between the continuous variables Z_g and Θ^* is given by

$$\rho(Z_g,\Theta) = \frac{a_g^* \, \sigma(\Theta^*)}{\{1 + [a_g^* \, \sigma(\Theta^*)]^2\}^{.5}} , \tag{2.2.21}$$

which is identically the correlation between the continuous variables Z_g and Θ. The scale of ability has no effect on the correlation. When a_g^* and b_g^* are known, that is, after $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$ have been prescribed, the point of dichotomization on the continuous variable Z_g, γ_g, is provided by

$$\gamma_g = \frac{b_g^* - \mu(\Theta^*)}{[(a_g^*)^{-2} + \sigma^2(\Theta^*)]^{.5}} , \tag{2.2.22}$$

which is also invariant with respect to a change of scale in ability.

In the general case the least squares estimators of ability are merely the means of Θ^* given the two possible outcomes: the mean of Θ^* given a correct answer to free-response item g, $\mu(\Theta^* \mid u_g = 1)$; and the mean of Θ^* given an incorrect answer to free-response item g, $\mu(\Theta^* \mid u_g = 0)$.

General Case: The Mean of Θ^* Given a Correct Answer to Item g, $\mu(\Theta^* \mid u_g = 1)$. This least squares estimator of ability is conveniently provided by

$$\begin{aligned} \mu(\Theta^* \mid u_g = 1) &= \mu(\Theta^*) + \rho(Z_g,\Theta) \, \sigma(\Theta^*) \, \mu(Z_g \mid u_g = 1) \\ &= \mu(\Theta^*) + \rho(Z_g,\Theta) \, \sigma(\Theta^*) \, \frac{\phi(\gamma_g)}{P_g} . \end{aligned} \tag{2.2.23}$$

In (2.2.23), $\mu(\Theta^*)$ and $\sigma(\Theta^*)$ are the prescribed mean and standard deviation of the continuous variable Θ^*; $\rho(Z_g,\Theta)$ is the correlation between the continuous variables Z_g and Θ^* as provided by (2.2.21); $\mu(Z_g \mid u_g = 1)$ is the mean of Z_g given a correct answer to item g as given by (2.2.4); γ_g is the point of dichotomization on the continuous variable Z_g as provided by (2.2.22); $\phi(\gamma_g)$ is the density evaluated at this point of dichotomization as obtained from (2.2.5); and P_g is the unconditional probability of a correct answer to item g as provided by (2.2.1). The values for $\rho(Z_g,\Theta)$, $\mu(Z_g \mid u_g = 1)$, γ_g, $\phi(\gamma_g)$, and P_g remain invariant under a linear transformation of scale on the continuous variable of ability.

Example 2.2.9. In the illustration that will be provided in Chapter 8, it is deemed desirable to obtain ability estimates that will have a mean of 100 and a standard deviation of 20. If the precision of these estimates is constant for all individuals at the termination of testing and comparable to that achieved with a test reliability, as will be defined and described in Chapter 7, of .90, then the mean, $\mu(\Theta^*)$, must be prescribed as 100 and the variance, $\sigma^2(\Theta^*)$, must be prescribed as 444.4444. The standard deviation, $\sigma(\Theta^*)$, must then be 21.0819.

In Example 2.1.1, it was found that a_g for item g as depicted in Figure 2.1.1 is 2.0647. When a_g^* of (2.2.17) is evaluated given the values of 2.0647 for a_g and 21.0819 for $\sigma(\Theta^*)$, a_g^* is found to be .09794. When $\rho(Z_g, \Theta)$ of (2.2.21) is evaluated with these values of a_g^* and $\sigma(\Theta^*)$, $\rho(Z_g,\Theta)$ is found to be .90 (the identical value that was obtained for $\rho(Z_g,\Theta)$ through the use of (2.2.10) in Example 2.2.7). Evaluating b_g^* of (2.2.18) given the values $-.375$, as obtained from Example 2.1.1, for b_g, 100 and 21.0819 as prescribed for $\mu(\Theta^*)$ and $\sigma(\Theta^*)$, respectively, b_g^* is found to be 92.0943. In obtaining γ_g from (2.2.22) given the values of .09794 for a_g^*, 92.0943 for b_g^*, 100 for $\mu(\Theta^*)$, and 444.4444 for $\sigma^2(\Theta^*)$, γ_g is found to be $-.3375$ (the identical value that was obtained for γ_g through the use of (2.2.2) in Example 2.2.1). Since γ_g is $-.3375$, $\phi(\gamma_g)$ and P_g have already been determined. Given this value for γ_g, $\phi(\gamma_g)$ was found to be .3769 in Example 2.2.3; and P_g was found to be .6321 in Example 2.2.1. Since $\phi(\gamma_g)$ and P_g are both invariant under a linear transformation of the scale of the continuous variable of ability, $\mu(Z_g \mid u_g = 1)$ as determined by (2.2.4) must also be. The mean of Z_g given a correct answer to item g, $\mu_g(Z_g \mid u_g = 1)$, was found to be .5963 in Example 2.2.3.

The least squares estimator of ability, $\mu(\Theta^* \mid u_g = 1)$, can now be evaluated through the use of (2.2.23). In this situation for this item g, it is known that $\mu(\Theta^*)$ is 100, $\rho(Z_g,\Theta)$ is .90, $\sigma(\Theta^*)$ is 21.0819, and $\mu(Z_g \mid u_g = 1)$, $\phi(\gamma_g)$ divided by P_g, is .5963. When $\mu(\Theta^* \mid u_g = 1)$ of (2.2.23) is evaluated given these values, $\mu(\Theta^* \mid u_g = 1)$ is found to be 111.3140. The value 111.3140 represents the mean of the distribution depicted in Figure 2.2.1 after a linear transformation of Θ into Θ^*. This transformation can be effected pictorially by merely changing the values along the abscissa in Figure 2.2.1 from -3.0 to 36.7543, from 0 to 100, and from $+3.0$ to 163.2457. With this transformation of the scale of ability, the distribution of Θ^* given u_g equal to *one*, $\Theta^*(u_g = 1)$, is analogous to the distribution of ability after incidental selection on Θ^* resulting from explicit selection on the continuous variable Z_g where γ_g is the cut score on the continuous variable Z_g.

In the psychological literature on personnel selection, an early occurrence of a mathematical expression closely related to that in (2.2.23) was due to Brogden (1949). In this article, the continuous variable comparable to Θ^* is a criterion measure of productivity, Y, rendered in a dollar metric with a standard deviation, $\sigma(Y)$; and the predictor of this criterion measure is a a continuous variable Z with a cut score γ, and a selection ratio P. The validity coefficient, $\rho(Z, Y)$, is the correlation between the predictor, Z, and the criterion measure, Y. Hence, the mean of Y given explicit selection on Z, $\mu(Y \mid \text{selection on } Z)$, is

$$\mu(Y \mid \text{selection on } Z) = \mu(Y) + \rho(Z,Y)\, \sigma(Y)\, \frac{\phi(\gamma)}{P}, \tag{2.2.24}$$

which is merely (2.2.23) interpreted into the context of personnel selection. The output of (2.2.24) is also the dollar productivity per individual selected because of the nature of the criterion variable Y. The unconditional mean on the criterion variable, $\mu(Y)$, is identically the mean of the continuous variable Y or the dollar productivity per individual under random selection. Hence, the gain in dollar productivity per individual selected over the situation where selection is at random, $\Delta\mu(Y \mid \text{selection on } Z)$, is given by

$$\Delta\mu(Y \mid \text{selection on } Z) = \rho(Z,Y)\, \sigma(Y)\, \frac{\phi(\gamma)}{P}, \tag{2.2.25}$$

into which the cost of testing can be incorporated. The cost of testing is given by the cost per individual examinee, C (a constant), divided by P, the selection ratio. Thus, the gain in dollar productivity per individual selected adjusted for the cost of testing, $\Delta\mu(Y \mid \text{selection on } Z, \text{cost of testing})$ is

$$\Delta\mu(Y \mid \text{selection on } Z, \text{cost of testing}) = \rho(Z,Y)\,\sigma(Y)\,\frac{\phi(\gamma)}{P} - \frac{C}{P}, \tag{2.2.26}$$

because random selection does not entail a cost for testing.

The utility of selection on the continuous variable Z adjusted for the cost of testing, U(selection on Z, cost of testing), is defined as the gain in the dollar value of productivity adjusted for the cost of testing given those individuals selected on the continuous variable Z. This utility is expressed as the number of those individuals selected on the continuous variable Z, N(selection on Z), multiplied by the output of (2.2.26), the gain per individual in the dollar value of productivity adjusted for the cost of testing. The number of individuals selected on the continuous variable Z, N(selection on Z), is obtained from

$$N(\text{selection on } Z) = PN, \tag{2.2.27}$$

where P is the selection ratio and N is the total number of individuals tested. Thus, the expression

$$\begin{aligned} &U(\text{selection on } Z, \text{cost of testing}) \\ &\qquad = N(\text{selection on } Z)\,\Delta\mu(Y \mid \text{selection on } Z, \text{cost of testing}) \end{aligned} \tag{2.2.28}$$

yields the gain in the dollar value of productivity adjusted for the cost of testing given those individuals selected on the predictor, the continuous variable Z, where γ is the cut score and, thus, P is the selection ratio.

General Case: The Mean of Θ^* Given an Incorrect Answer to Item g, $\mu(\Theta^* \mid u_g = 0)$. This least squares estimator of ability is conveniently provided by

$$\begin{aligned} \mu(\Theta^* \mid u_g = 0) &= \mu(\Theta^*) + \rho(Z_g,\Theta)\,\sigma(\Theta^*)\,\mu(Z_g \mid u_g = 0) \\ &= \mu(\Theta^*) - \rho(Z_g,\Theta)\,\sigma(\Theta^*)\,\frac{\phi(\gamma_g)}{Q_g}. \end{aligned} \tag{2.2.29}$$

In (2.2.29), $\mu(\Theta^*)$ and $\sigma(\Theta^*)$ are the prescribed mean and standard deviation of the continuous variable Θ^*; $\rho(Z_g,\Theta)$ is the correlation between the continuous variables Z_g and Θ^* as given by (2.2.21); $\mu(Z_g \mid u_g = 0)$ is the mean of Z_g given an incorrect answer to item g as provided by (2.2.6); γ_g is the point of dichotomization on the continuous variable Z_g as given by (2.2.22); $\phi(\gamma_g)$ is the density evaluated at this point of dichotomization as obtained from (2.2.5); and Q_g is the unconditional probability of an incorrect answer to item g as provided by (2.2.3). The values for $\rho(Z_g,\Theta)$, $\mu(Z_g \mid u_g = 0)$, γ_g, $\phi(\gamma_g)$, and P_g remain invariant under a linear transformation of the scale of the continuous variable of ability.

Example 2.2.10. As in the case of Example 2.2.9, it is viewed as desirable to obtain ability estimates that at the termination of testing will have a mean of 100 and a standard deviation of 20. These respective values can be obtained by terminating the tests for all individuals at a level of precision comparable to that achieved with a tailored test reliability of .90 and prescribing the mean of Θ^*, $\mu(\Theta^*)$, as 100 and the variance of Θ^*, $\sigma^2(\Theta^*)$, as 444.4444 or the standard deviation of Θ^*, $\sigma(\Theta^*)$, as 21.0819.

In Example 2.2.9, $\rho(Z_g,\Theta)$, γ_g, and $\phi(\gamma_g)$ were found to be .90, $-.3375$, and .3769, respectively, for item g as depicted in Figure 2.1.1. Since γ_g is invariant under a linear transformation of scale of the continuous variable of ability, so must Q_g as determined by (2.2.3) be invariant. The unconditional probability of an incorrect answer to item g, Q_g, was found to be .3679 in Example 2.2.2. The mean of Z_g given an incorrect answer to item g is also invariant under a linear transformation of scale of the continuous variable of ability. This invariance necessarily follows since $\phi(\gamma_g)$ and Q_g in (2.2.6) are invariant under such a transformation. In Example 2.2.4, $\mu(Z_g \mid u_g = 0)$ was determined to be -1.0245.

The least squares estimator, $\mu(\Theta^* \mid u_g = 0)$, can now be evaluated through the use of (2.2.29). For this situation and for this item g, it is known that $\mu(\Theta^*)$ is 100, $\rho(Z_g,\Theta)$ is .90, $\sigma(\Theta^*)$ is 21.0819, and $\mu(Z_g \mid u_g = 0)$, the negative of $\phi(\gamma_g)$ divided by Q_g, is -1.0245. When $\mu(\Theta^* \mid u_g = 0)$ of (2.2.29) is evaluated given these values, $\mu(\Theta^* \mid u_g = 0)$ is found to be 80.5614. The value 80.5614 represents the mean of the distribution depicted in Figure 2.2.2 after a transformation of Θ into Θ^*. This transformation can be effected pictorially by merely changing the values along the abscissa in Figure 2.2.2 from -3.0 to 36.7543, from 0 to 100, and from $+3.0$ to 163.2457. The distribution of Θ^* given an incorrect answer to item g, $\Theta^*(u_g = 0)$, is analogous to the distribution of Θ^* after incidental rejection on Θ^* resulting from explicit rejection on the continuous variable Z_g where γ_g is the cut score on the continuous variable Z_g.

In tailored testing, the general case of ability estimation is the more interesting case. As will be seen in later chapters, this case allows the sequential updating of ability estimates as more items are added to a tailored test. The mean and variance of the distribution of ability change as each answer to a tailored test is scored; and it is the general case that produces estimators of ability under these changes. The specific case is only appropriate for a tailored test consisting of a single item. This case cannot accomodate the changes in the mean and variance of the distribution of ability that occur with the sequential scoring of items in a tailored test.

2.3. Mathematical Proofs

The convenient expressions that are derived in this section were presented and numerically illustrated earlier in Section 2.2. The mathematical proofs for these expressions as contained in this section may be omitted by the reader who is seeking a general understanding. The omission of this section will not result in a loss of continuity.

At this juncture, a brief review of the bivariate normal distribution is in order. A helpful background for this review can be found in Mood and Graybill (1963). It will be shown that the joint distribution of the continuous variables Z_g and Θ is bivariate normal. This relationship will be established from the fundamental assumptions of the basic model for a single common factor presented in Chapter 1. Bivariate normality facilitates the derivation of many useful mathematical forms.

Bivariate Normal Distribution

In the present instance, the joint density function for the bivariate normal distribution is given by

$$\phi(\zeta_g,\theta) = (2\pi)^{-1}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left\{-.5\left[\frac{\zeta_g^2 - 2\rho(Z_g,\Theta)\,\zeta_g\theta + \theta^2}{1-\rho^2(Z_g,\Theta)}\right]\right\}, \tag{2.3.1}$$

where a density function is by definition a relative frequency. In the case of a joint density function, as in (2.3.1), its integral over the plane must be unity,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\zeta_g,\theta)\,d\zeta_g\,d\theta = 1. \tag{2.3.2}$$

This equality may be proved by completing the square on ζ_g in the exponent of (2.3.1), which yields

$$\begin{aligned}&\phi(\zeta_g,\theta)\\ &= (2\pi)^{-1}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left(\frac{-.5}{1-\rho^2(Z_g,\Theta)}\left\{[\zeta_g-\rho(Z_g,\Theta)\theta]^2 + [1-\rho^2(Z_g,\Theta)]\theta^2\right\}\right).\end{aligned} \tag{2.3.3}$$

After rearranging the exponent in (2.3.3), it is found that

$$\phi(\zeta_g,\theta) = (2\pi)^{-1}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left(-.5\left\{\frac{\zeta_g-\rho(Z_g,\Theta)\theta}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2 - .5\theta^2\right), \tag{2.3.4}$$

which when integrated over the plane may be represented as,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(2\pi)^{-1}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left(-.5\left\{\frac{\zeta_g-\rho(Z_g,\Theta)\,\theta}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2 - .5\,\theta^2\right)d\zeta_g\,d\theta. \tag{2.3.5}$$

It is to be noted that $\tilde{\zeta}_g(\theta)$ of (2.1.10) can now be substituted into the exponent of (2.3.5). Further, since the derivative of $\tilde{\zeta}_g(\theta)$ with respect to ζ_g is

$$\frac{d\tilde{\zeta}_g(\theta)}{d\zeta_g} = \frac{d\dfrac{\zeta_g-\rho(Z_g,\Theta)\theta}{[1-\rho^2(Z_g,\Theta)]^{.5}}}{d\zeta_g} = [1-\rho^2(Z_g,\Theta)]^{-.5}, \tag{2.3.6}$$

it is known that the differential of $\tilde{\zeta}_g(\theta)$ in relation to that of ζ_g is

$$d\tilde{\zeta}_g(\theta) = d\zeta_g\,[1-\rho^2(Z_g,\Theta)]^{-.5}, \tag{2.3.7}$$

which can also be substituted into (2.3.5). These substitutions allow one to write the integral as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5\, [\tilde{\zeta}_g(\theta)]^2\}\, d\tilde{\zeta}_g(\theta)\, (2\pi)^{-.5} \exp(-.5\theta^2)\, d\theta, \tag{2.3.8}$$

which can be expressed as the product of two integrals

$$\int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5\, [\tilde{\zeta}_g(\theta)]^2\}\, d\tilde{\zeta}_g(\theta) \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp(-.5\theta^2)\, d\theta \tag{2.3.9}$$

because it is known that the innermost integral in (2.3.8) is a constant during integration with respect to θ. This integral, now on the left side of (2.3.9), is the cumulative normal distribution function of (2.1.18). In this case, it represents the entire area under the normal curve, which is known from previous discussion to be unity. The integral on the right side of (2.3.9) is also a cumulative normal distribution function, in this case, that for the continuous variable Θ. Because Θ is assumed to be normally distributed with a mean of zero and a variance of unity, this integral is also known to be unity. Because the product of the integrals is unity, (2.3.2) has been proved.

The density of the unconditional or marginal distribution of Θ is by definition

$$\phi(\theta) = \int_{-\infty}^{\infty} \phi(\zeta_g,\theta)\, d\zeta_g, \tag{2.3.10}$$

where a substitution from (2.3.4) yields

$$\phi(\theta) = \int_{-\infty}^{\infty} (2\pi)^{-1}\, [1 - \rho^2(Z_g,\Theta)]^{-.5} \exp\left(-.5\left\{\frac{\zeta_g - \rho(Z_g,\Theta)\theta}{[1 - \rho^2(Z_g,\Theta)]^{.5}}\right\}^2 - .5\theta^2\right) d\zeta_g. \tag{2.3.11}$$

Again it is noted that $\tilde{\zeta}_g(\theta)$ of (2.1.10) can be substituted for the expression in braces in the exponent. To obtain the integral with respect to $\tilde{\zeta}_g(\theta)$, a substitution from (2.3.7) is also necessary. These substitutions lead to

$$\phi(\theta) = (2\pi)^{-.5} \exp(-.5\theta^2) \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5\, [\tilde{\zeta}_g(\theta)]^2\}\, d\tilde{\zeta}_g(\theta), \tag{2.3.12}$$

where it is known from previous discussion that the value of the integral is unity because the integral represents the cumulative normal distribution function for the total area under a normal curve. Thus the marginal density of Θ is

$$\phi(\theta) = (2\pi)^{-.5} \exp(-.5\theta^2), \tag{2.3.13}$$

or merely the density of the univariate normal distribution.

Notice that (2.3.4) after a substitution from (2.1.10) may be written as

$$\phi(\zeta_g,\theta) = [1 - \rho^2(Z_g,\Theta)]^{-.5}\, (2\pi)^{-.5} \exp\{-.5[\tilde{\zeta}_g(\theta)]^2\}\, (2\pi)^{-.5} \exp(-.5\theta^2), \tag{2.3.14}$$

the triple product of a scalar and two univariate normal densities. This scalar maintains a probability volume of unity under the normal bivariate surface when the conditional distributions decrease in effective range about either line of regression as the correlation, $\rho(Z_g,\Theta)$, transits in values from zero to unity. One of the densities, as can be seen from (1.1.3) and (1.1.6), represents the normal probability density for the homoscedastic error variable Δ_g while the other represents the univariate normal density for the continuous variable Θ. Notice that $\tilde{\zeta}_g(\Theta)$ of (2.1.10) is merely Δ_g converted to a variance of unity. The means of the Δ_g are already zero, as can be seen in (1.1.9). The fundamental assumptions of the model for a single common factor characterize the joint distribution of each Z_g and Θ as bivariate normal. If the regression of one random variable on another random variable is linear, and the marginal distribution of that random variable is defined as univariate normal, and the conditional distributions for fixed values of that variable are homoscedastic and univariate normal, then the joint distribution is necessarily bivariate normal. This relationship also means that the other marginal distribution is univariate normal and that the other conditional distributions for fixed values of the other marginal random variable are also homoscedastic and univariate normal.

The standardized distributions of Z_g given θ, the $\tilde{Z}_g(\theta)$, have been discussed in considerable detail with respect to their relationship to the item characteristic curve of the two-parameter normal ogive submodel.

Illustrations were provided in Figures 2.1.1 and 2.1.2. The exposition will now turn to the conditional distributions of Θ for fixed values of Z_g and the unconditional or marginal distribution of Z_g. This exposition will require some detail because of the special relationship between Z_g and the binary variable U_g.

Discussion is facilitated by completing the square on θ in the exponent of (2.3.1). This operation yields

$$\phi(\zeta_g,\theta)$$
$$=(2\pi)^{-1}\,[1 - \rho^2(Z_g,\Theta)]^{-.5}\exp\left(\frac{-.5}{1-\rho^2(Z_g,\Theta)}\left\{[\theta - \rho(Z_g,\Theta)\,\zeta_g]^2 + [1-\rho^2(Z_g,\Theta)]\,\zeta_g^2\right\}\right). \quad (2.3.15)$$

Equation (2.3.15) may be rearranged in the exponent as

$$\phi(\zeta_g,\theta) = (2\pi)^{-1}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left(-.5\left\{\frac{\theta-\rho(Z_g,\Theta)\,\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2 - .5\,\zeta_g^2\right), \quad (2.3.16)$$

which when integrated over the plane may be represented as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(2\pi)^{-.5}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left(-.5\left\{\frac{\theta-\rho(Z_g,\Theta)\,\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2\right)d\theta\,(2\pi)^{-.5}\exp(-.5\,\zeta_g^2)\,d\zeta_g. \quad (2.3.17)$$

Let $\bar{\theta}(\zeta_g)$ be defined as

$$\bar{\theta}(\zeta_g) = \frac{\theta-\rho(Z_g,\Theta)\,\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}, \quad (2.3.18)$$

where the similarity between $\bar{\zeta}_g(\theta)$ of (2.1.10) and $\bar{\theta}(\zeta_g)$ of (2.3.18) is apparent. Both expressions represent standardized scores from normal conditional distributions. In the case of (2.3.18), the conditional distributions are those of Θ given ζ_g. It may be noted that the derivative of $\bar{\theta}(\zeta_g)$ with respect to θ is

$$\frac{d\bar{\theta}(\zeta_g)}{d\theta} = \frac{d\,\dfrac{\theta-\rho(Z_g,\Theta)\,\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}}{d\theta} = [1-\rho^2(Z_g,\Theta)]^{-.5}. \quad (2.3.19)$$

Therefore, the differential of $\bar{\theta}(\zeta_g)$ in relation to that of θ is given by

$$d\bar{\theta}(\zeta_g) = d\theta\,[1-\rho^2(Z_g,\Theta)]^{-.5}. \quad (2.3.20)$$

Substitutions from (2.3.18) and (2.3.20) into (2.3.17) now allow one to write

$$\int_{-\infty}^{\infty}(2\pi)^{-.5}\exp\{-.5[\bar{\theta}(\zeta_g)]^2\}\,d\bar{\theta}(\zeta_g)\int_{-\infty}^{\infty}(2\pi)^{-.5}\exp(-.5\,\zeta_g^2)\,d\zeta_g, \quad (2.3.21)$$

the product of two integrals both known to have a value of unity. Thus, the integral over the plane or the probability volume under the bivariate surface as given in (2.3.2) can also be verified as unity from this perspective.

The density of the unconditional or marginal distribution of Z_g is by definition

$$\phi(\zeta_g) = \int_{-\infty}^{\infty}\phi(\zeta_g,\theta)\,d\theta, \quad (2.3.22)$$

where a substitution from (2.3.16) permits one to write

$$\phi(\zeta_g) = \int_{-\infty}^{\infty}(2\pi)^{-1}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left(-.5\left\{\frac{\theta-\rho(Z_g,\Theta)\,\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2 - .5\,\zeta_g^2\right)d\theta. \quad (2.3.23)$$

Substitutions from (2.3.18) and (2.3.20) into (2.3.23) lead to

$$\phi(\zeta_g) = (2\pi)^{-.5}\exp(-.5\,\zeta_g^2)\int_{-\infty}^{\infty}(2\pi)^{-.5}\exp\{-.5[\bar{\theta}(\zeta_g)]^2\}\,d\bar{\theta}(\zeta_g), \quad (2.3.24)$$

where the value of the integral is unity. As a result, it is found that

$$\phi(\zeta_g) = (2\pi)^{-.5} \exp(-.5\zeta_g^2), \tag{2.3.25}$$

indicating that the unconditional or marginal density of the continuous variable Z_g is merely the density of the univariate normal distribution.

Given the developments from (2.3.3) to (2.3.9) and from (2.3.15) to (2.3.21), it is apparent that the bivariate normal distribution can be portrayed as comprised of infinitely many conditional distributions of Z_g given the random variable Θ as well as comprised of infinitely many conditional distributions of Θ given the random variable Z_g. The former portrayal was presented in abbreviated fashion in Figure 2.1.1, while the latter is presented in an analogous fashion in Figure 2.3.1. In this figure, it is important to point out that the horizontal projection of the point of dichotomization, γ_g, across the bivariate normal distribution runs parallel to the conditional distributions of Θ given ζ_g. Hence, these distributions and thus their means and variances are undisturbed by the binary scoring of item g based on Z_g. As will later be seen, the means and variances of Θ given ζ_g are left undisturbed by another method of binary scoring for a multiple-choice item g that is determined by Z_g. These relationships will become important when selection theory is considered in Chapter 5. These relationships can be sharply contrasted with those presented in Figure 2.1.1 where the bisection of the $Z_g(\theta)$ based on the binary scoring of item g, the dichotomization of Z_g, gives rise to the item characteristic curve of Figure 2.1.2.

In Figures 2.1.1 and 2.3.1, the bivariate normal distribution is represented as consisting of conditional distributions with unit area that are homoscedastic and univariate normal. Though accurate, this representation is not the typical representation. The typical representation is displayed in Figure 2.3.2. In this figure, the bivariate normal distribution is represented by a configuration of ellipses, each arising from the display of points of equal joint density. This representation more closely resembles the sample counterpart of the bivariate normal distribution, the bivariate normal scatterplot. The apparent dissimilarity between Figure 2.3.2 and Figures 2.1.1 and 2.3.1 is easily resolved. The relative frequency within each conditional distribution is given by the univariate normal density function for the fixed value of the marginal variable. This relationship is made explicit in (2.3.12) for θ and in (2.3.24) for ζ_g. Thus, one may choose to take into account the relative frequencies or marginal

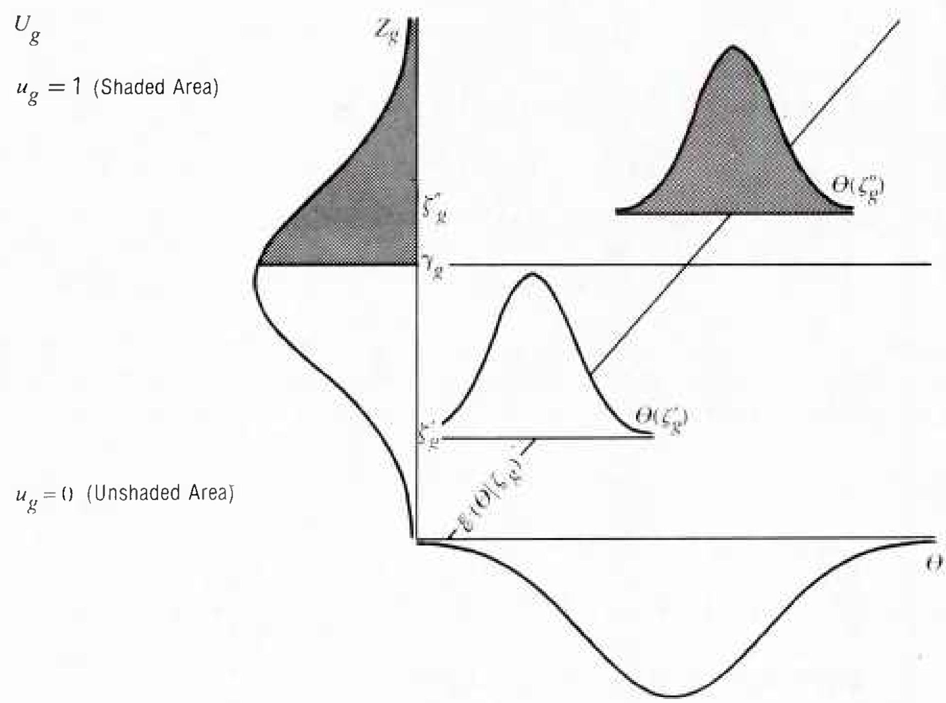

Figure 2.3.1 Hypothetical relations among the item continuum Z_g, the free-response, binary item U_g, the conditional distributions of Θ given ζ_g, and the latent trait Θ.

densities. In this type of representation, one arrives at the typical portrayal presented in Figure 2.3.2. The manner of portrayal of the bivariate normal distribution is thus optional. The choice of a particular manner depends then on the relationships that are to be explained.

In the discussion that follows, convenient expressions are derived for various terms. These derivations will begin with the defining relationships within the bivariate normal distribution. During this initial stage of derivation, the reader may find Figure 2.3.2 helpful in visualizing the particular defining relationship that serves as a basis for the derivation.

The Unconditional Probabilities for the Realizations of U_g

The binary random variable U_g can realize one of two possible values, either *one* or *zero*, indicating a correct or an incorrect answer, respectively, to free-response item g. Expressions will now be derived for the probabilities of these two possible outcomes.

The Probability of A Correct Answer to Item g, $\Pr(u_g = 1)$. This unconditional probability is more commonly referred to as the p-value for item g; however, in this context it shall be designated as P_g. By definition it is known that

$$P_g = \Pr(u_g = 1) = \Pr(Z_g \geqslant \gamma_g) = \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \phi(\zeta_g, \theta)\, d\theta\, d\zeta_g, \qquad (2.3.26)$$

where the rightmost term is the integral over the plane from γ_g to positive infinity on Z_g and from negative infinity to positive infinity on Θ. It should be noted that γ_g, because of (2.1.13), can be conveniently expressed as

$$\gamma_g = \rho(Z_g, \Theta)\, b_g, \qquad (2.3.27)$$

where $\rho(Z_g, \Theta)$, because of (2.1.12), can be conveniently expressed as

$$\rho(Z_g, \Theta) = \frac{a_g}{(1 + a_g^2)^{.5}}. \qquad (2.3.28)$$

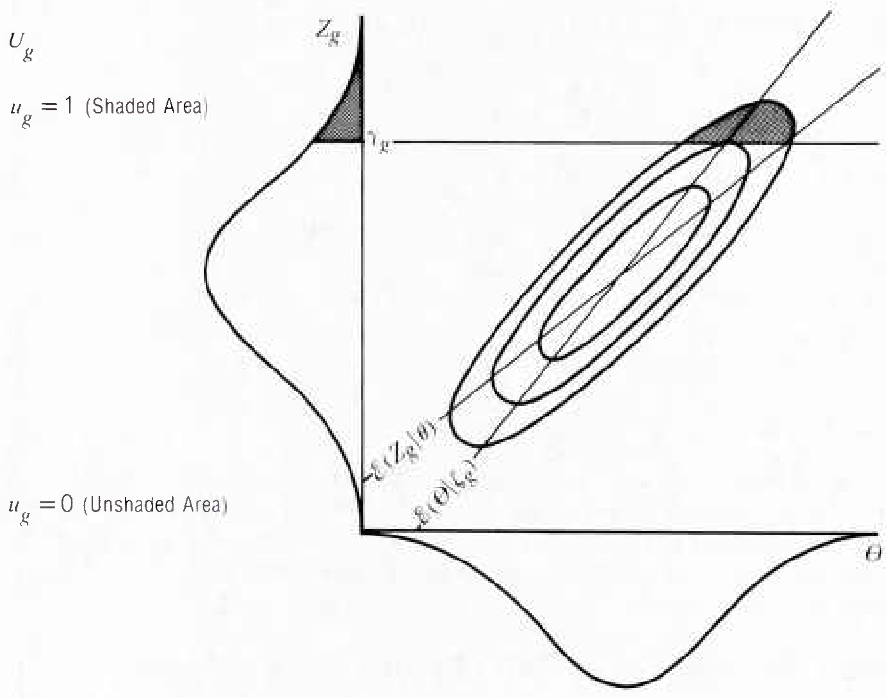

Figure 2.3.2 A typical display of the bivariate normal distribution (including the univariate normal marginal distributions of Z_g and Θ).

After a substitution from (2.3.16) into the rightmost term in (2.3.26), the integral in (2.3.26) can be expressed as

$$P_g = \int_{\gamma_g}^{\infty} (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5}[1-\rho^2(Z_g,\Theta)]^{-.5} \exp\left(-.5\left\{\frac{\theta-\rho(Z_g,\Theta)\,\zeta_g}{[1-\rho^2(Z_g,\theta)]^{.5}}\right\}^2\right) d\theta, \tag{2.3.29}$$

where substitutions from (2.3.18) and (2.3.20) yield

$$P_g = \int_{\gamma_g}^{\infty} (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g). \tag{2.3.30}$$

Again, it is to be noted that the rightmost integral in (2.3.30) is unity, which allows the writing of

$$P_g = \int_{\gamma_g}^{\infty} (2\pi)^{-.5} \exp(-.5\zeta_g^2) d\zeta_g = \int_{-\infty}^{-\gamma_g} (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g = \Phi(-\gamma_g), \tag{2.3.31}$$

where Φ (*) is by definition the cumulative normal distribution function. The equality of integrals in (2.3.31) is due, again, to the symmetry of the normal distribution. The probability of a correct answer to item g, P_g, is pictorially represented by the shaded area in the marginal distribution of Z_g in Figure 2.1.1. This representation is possible because the truncated marginal distribution, as can be seen in the developments from (2.3.29) to (2.3.31), merely summarizes the truncated joint distribution for Z_g.

The Probability of An Incorrect Answer to Item g, $\Pr(u_g = 0)$. As will become evident, this unconditional probability is the complement of P_g or $(1 - P_g)$, because the area under the normal curve is unity. Let this probability be designated as Q_g. By definition then, it is known that

$$Q_g = \Pr(u_g = 0) = \Pr(Z_g < \gamma_g) = \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \phi(\zeta_g, \theta)\, d\theta\, d\zeta_g, \tag{2.3.32}$$

where the rightmost term is the integral over the plane from negative infinity to γ_g on Z_g and from negative infinity to positive infinity on Θ. After a substitution from (2.3.16) into the rightmost term in (2.3.32), this integral can be expressed as

$$Q_g = \int_{-\infty}^{\gamma_g} (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5}[1-\rho^2(Z_g,\Theta)]^{-.5} \exp\left(-.5\left\{\frac{\theta-\rho(Z_g,\Theta)\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2\right) d\theta, \tag{2.3.33}$$

where substitutions from (2.3.18) and (2.3.20) provide

$$Q_g = \int_{-\infty}^{\gamma_g} (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g). \tag{2.3.34}$$

Again, it is to be noted that the rightmost integral is unity, which allows the writing of

$$Q_g = \int_{-\infty}^{\gamma_g} (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g = \Phi(\gamma_g), \tag{2.3.35}$$

where Φ(*) is again the cumulative normal distribution function.

It is now apparent from a review of (2.3.31) and (2.3.35) that P_g and Q_g are complementary terms. These terms account for the entire area in the normal distribution where this area is unity. The probability of an incorrect answer to item g, Q_g, is portrayed in Figure 2.1.1 as the unshaded area in the marginal distribution of Z_g. This representation is possible because, as can be seen in the developments from (2.3.33) to (2.3.35), the truncated marginal distribution merely summarizes the truncated joint distribution for Z_g.

The Conditional Means of Z_g Given the Realizations of U_g

The mean of Z_g can assume only one of two possible values depending on whether item g was answered correctly or incorrectly. Expressions will now be derived for these means or expected values for the two possible outcomes.

The Mean of Z_g Given a Correct Answer to Item g, $\mu(Z_g \mid u_g = 1)$. This mean is defined by

$$\mu(Z_g \mid u_g = 1) = \mathscr{E}(Z_g \mid u_g = 1) = \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g \, \phi^*(\zeta_g, \theta) \, d\theta \, d\zeta_g, \tag{2.3.36}$$

where $\phi^*(\zeta_g, \theta)$ is the joint density function for this particular double integral. This joint density function is given by

$$\phi^*(\zeta_g, \theta) = \frac{\phi(\zeta_g, \theta)}{P_g}, \tag{2.3.37}$$

where it is necessary to show that the probability integral

$$\int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \phi^*(\zeta_g, \theta) \, d\theta \, d\zeta_g \tag{2.3.38}$$

equals unity in order to verify that (2.3.37) is the joint density function. Since the reciprocal of P_g is a constant with respect to the integration of (2.3.38), a substitution from (2.3.37) into (2.3.38) yields

$$\frac{1}{P_g} \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \phi(\zeta_g, \theta) \, d\theta \, d\zeta_g, \tag{2.3.39}$$

which may be seen to equal unity because of (2.3.26). The substitution of (2.3.37) into (2.3.36) leads to

$$\mu(Z_g \mid u_g = 1) = \frac{1}{P_g} \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g \, \phi(\zeta_g, \theta) \, d\theta \, d\zeta_g, \tag{2.3.40}$$

which follows from the defining relationship stated in (2.3.36). After substituting from (2.3.16) into (2.3.40) and proceeding with the integration, one may write

$$\mu(Z_g \mid u_g = 1)$$

$$= \frac{1}{P_g} \int_{\gamma_g}^{\infty} \zeta_g \, (2\pi)^{-.5} \exp(-.5\zeta_g^2) \, d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5} \, [1 - \rho^2(Z_g, \Theta)]^{-.5} \exp\left(-.5 \left\{ \frac{\theta - \rho(Z_g, \Theta)\zeta_g}{[1 - \rho^2(Z_g, \Theta)]^{.5}} \right\}^2 \right) d\theta \tag{2.3.41}$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\mu(Z_g \mid u_g = 1) = \frac{1}{P_g} \int_{\gamma_g}^{\infty} \zeta_g \, (2\pi)^{-.5} \exp(-.5\zeta_g^2) \, d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\} \, d\tilde{\theta}(\zeta_g). \tag{2.3.42}$$

Again, it is to be noted that the rightmost integral in (2.3.42) is unity, which allows one to write, merely,

$$\mu(Z_g \mid u_g = 1) = \frac{1}{P_g} \int_{\gamma_g}^{\infty} \zeta_g (2\pi)^{-.5} \exp(-.5\zeta_g^2) \, d\zeta_g, \tag{2.3.43}$$

where the antiderivative of the integrand, which is required in subsequent development, is known to be

$$-\phi(\zeta_g) = -(2\pi)^{-.5} \exp(-.5\zeta_g^2). \tag{2.3.44}$$

That (2.3.44) provides the antiderivative of the integrand in (2.3.43) can be verified. One can take the derivative of (2.3.44) with respect to the variable of integration. This derivative must then equal the integrand. By way of verification, the derivative of (2.3.44) with respect to the variable of integration is found to be

$$\frac{d\,[-\phi(\zeta_g)]}{d\zeta_g} = \frac{d\,[-(2\pi)^{-.5} \exp(-.5\zeta_g^2)]}{d\zeta_g} = -(2\pi)^{-.5} \frac{d \exp(-.5\zeta_g^2)}{d\zeta_g}$$

$$= -(2\pi)^{-.5} \exp(-.5\zeta_g^2) \frac{d\,(-.5\zeta_g^2)}{d\zeta_g} = \zeta_g \, (2\pi)^{-.5} \exp(-.5\zeta_g^2), \tag{2.3.45}$$

where the rightmost equality is indeed the integrand in (2.3.43).

A solution for an integral usually consists of evaluating the antiderivative of the integrand at the upper and lower limits of integration and obtaining the simple difference. But the upper limit in (2.3.43) is positively infinite. Thus, the limit of the integral must be evaluated as b, the upper limit, approaches infinity:

$$\int_{\gamma_g}^{\infty} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g = \lim_{b\to\infty}\int_{\gamma_g}^{b} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g = \lim_{b\to\infty}\,[-\phi(b)+\phi(\gamma_g)]. \quad (2.3.46)$$

Because the antiderivative at b, $-\phi(b)$, approaches zero as b becomes positively infinite, it is known that

$$\int_{\gamma_g}^{\infty} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g = \phi(\gamma_g), \quad (2.3.47)$$

where a substitution from (2.3.47) into (2.3.43) provides the desired result

$$\mu(Z_g \mid u_g = 1) = \frac{\phi(\gamma_g)}{P_g}. \quad (2.3.48)$$

Thus, the mean of Z_g given a correct answer to item g, $\mu(Z_g \mid u_g = 1)$, is simply the value of the density at γ_g in the standard normal distribution, $\phi(\gamma_g)$, divided by the unconditional probability of a correct answer to item g, P_g.

The Mean of Z_g Given an Incorrect Answer to Item g, $\mu(Z_g \mid u_g = 0)$. This mean is defined by

$$\mu\,(Z_g \mid u_g = 0) = \mathscr{E}(Z_g \mid u_g = 0) = \int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \zeta_g\,\phi^{\dagger}(\zeta_g,\theta)\,d\theta\,d\zeta_g, \quad (2.3.49)$$

where $\phi^{\dagger}(\zeta_g,\theta)$ is the joint density function for this particular double integral. This joint density function is given by

$$\phi^{\dagger}(\zeta_g,\theta) = \frac{\phi(\zeta_g,\theta)}{Q_g}, \quad (2.3.50)$$

where it is necessary to show that the double integral

$$\int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \phi^{\dagger}(\zeta_g,\theta)\,d\theta\,d\zeta_g \quad (2.3.51)$$

equals unity in order to verify that (2.3.50) provides the joint density function. Since the reciprocal of Q_g is a constant with respect to the integration of (2.3.51), a substitution from (2.3.50) into (2.3.51) provides

$$\frac{1}{Q_g}\int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \phi(\zeta_g,\theta)\,d\theta\,d\zeta_g, \quad (2.3.52)$$

which may be seen to equal unity because of (2.3.32). The substitution of (2.3.50) into (2.3.49) leads to

$$\mu(Z_g \mid u_g = 0) = \frac{1}{Q_g}\int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \zeta_g\,\phi(\zeta_g,\theta)\,d\theta\,d\zeta_g. \quad (2.3.53)$$

After substituting from (2.3.16) into (2.3.53) and continuing with the integration, one may write

$$\mu(Z_g \mid u_g = 0)$$
$$= \frac{1}{Q_g}\int_{-\infty}^{\gamma_g} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5}\,[1-\rho^2(Z_g,\Theta)]^{-.5}\exp\left(-.5\left\{\frac{\theta-\rho(Z_g,\Theta)\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2\right)d\theta, \quad (2.3.54)$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\mu(Z_g \mid u_g = 0) = \frac{1}{Q_g}\int_{-\infty}^{\gamma_g} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5}\exp\{-.5[\bar\theta(\zeta_g)]^2\}\,d\bar\theta(\zeta_g). \quad (2.3.55)$$

Again it is to be noted that the rightmost integral is unity, which allows one to write

$$\mu(Z_g \mid u_g = 0) = \frac{1}{Q_g}\int_{-\infty}^{\gamma_g} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g, \quad (2.3.56)$$

where the antiderivative of the integrand in (2.3.56) is given in (2.3.44) and verified as such in (2.3.45). A solution for an integral usually consists of evaluating the antiderivative at the upper and lower limits of integration and obtaining the simple difference. But the lower limit is negatively infinite. Thus, the limit of the integral must be evaluated as a, the lower limit, approaches negative infinity:

$$\begin{aligned}\int_{-\infty}^{\gamma_g} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)d\zeta_g &= \lim_{a\to-\infty}\int_a^{\gamma_g} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g \\ &= \lim_{a\to-\infty}\,[-\phi(\gamma_g) + \phi(a)].\end{aligned} \tag{2.3.57}$$

Because the antiderivative at $a, -\phi(a)$, approaches zero as a becomes negatively infinite, it is known that

$$\int_{-\infty}^{\gamma_g} \zeta_g\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g = -\,\phi(\gamma_g) \tag{2.3.58}$$

where a substitution from (2.3.58) into (2.3.56) provides the desired result

$$\mu(Z_g \mid u_g = 0) = -\,\frac{\phi(\gamma_g)}{Q_g}. \tag{2.3.59}$$

Thus, the mean of Z_g given an incorrect answer to item g, $\mu(Z_g \mid u_g = 0)$, is simply the negative of the value of the density at γ_g in the standard normal distribution, $-\phi(\gamma_g)$, divided by the unconditional probability of an incorrect answer to item g, Q_g.

The Conditional Variances of Z_g Given the Realizations of U_g

The variance of Z_g can assume only one of two possible values depending on whether the item was answered correctly or incorrectly. Expressions will now be derived for these variances given the two possible outcomes.

The Variance of Z_g Given a Correct Answer to Item g, $\sigma^2(Z_g \mid u_g = 1)$. In solving for the variance of Z_g given a correct answer to item g, an expression is found for $\mathscr{E}(Z_g^2 \mid u_g = 1)$. This quantity represents the expected value of the sum of squared deviations from a mean that was appropriate before the truncation of Z_g at γ_g. Later, an expression is obtained for the desired variance, $\sigma^2(Z_g \mid u_g = 1)$ or the expected value of the sum of the squared deviations about the mean subsequent to this truncation, through

$$\sigma^2(Z_g \mid u_g = 1) = \mathscr{E}(Z_g^2 \mid u_g = 1) - \mu^2(Z_g \mid u_g = 1), \tag{2.3.60}$$

which is an explicit solution for $\sigma^2(Z_g \mid u_g = 1)$ given the identity

$$\mathscr{E}(Z_g^2 \mid u_g = 1) = \sigma^2(Z_g \mid u_g = 1) + \mu^2(Z_g \mid u_g = 1). \tag{2.3.61}$$

Proceeding with the solution for $\mathscr{E}(Z_g^2 \mid u_g = 1)$, it is known by definition that

$$\mathscr{E}(Z_g^2 \mid u_g = 1) = \int_{\gamma_g}^{\infty}\int_{-\infty}^{\infty} \zeta_g^2\,\phi^*(\zeta_g,\theta)\,d\theta\,d\zeta_g, \tag{2.3.62}$$

where $\phi^*(\zeta_g,\theta)$ is given in (2.3.37) and verified as the joint density function in the discussion surrounding (2.3.39). A substitution from (2.3.37) into (2.3.62) allows one to write

$$\mathscr{E}(Z_g^2 \mid u_g = 1) = \frac{1}{P_g}\int_{\gamma_g}^{\infty}\int_{-\infty}^{\infty} \zeta_g^2\,\phi(\zeta_g,\theta)\,d\theta\,d\zeta_g. \tag{2.3.63}$$

After substituting from (2.3.16) into (2.3.63) and continuing with the integration, one may write

$$\begin{aligned}&\mathscr{E}(Z_g^2 \mid u_g = 1) \\ &= \frac{1}{P_g}\int_{\gamma_g}^{\infty} \zeta_g^2\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g \int_{-\infty}^{\infty}(2\pi)^{-.5}\,[1-\rho^2(Z_g,\theta)]^{-.5}\exp\left(-.5\left\{\frac{\theta-\rho(Z_g,\Theta)\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}}\right\}^2\right)d\theta\end{aligned} \tag{2.3.64}$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\mathscr{E}(Z_g^2 \mid u_g = 1) = \frac{1}{P_g}\int_{\gamma_g}^{\infty} \zeta_g^2 (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g). \quad (2.3.65)$$

Again, it is to be noted that the rightmost integral in (2.3.65) is unity, which allows the writing of

$$\mathscr{E}(Z_g^2 \mid u_g = 1) = \frac{1}{P_g}\int_{\gamma_g}^{\infty} \zeta_g^2 (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g, \quad (2.3.66)$$

where it is customary to solve for this integral through the method of integration by parts. One begins by defining r and s as

$$r = \zeta_g \quad (2.3.67)$$

and

$$s = -\phi(\zeta_g) = -(2\pi)^{-.5} \exp(-.5\, \zeta_g^2). \quad (2.3.68)$$

Notice that the derivative of r with respect to ζ_g is unity:

$$\frac{dr}{d\zeta_g} = \frac{d\zeta_g}{d\zeta_g} = 1. \quad (2.3.69)$$

Thus the differential of r in relation to that of ζ_g is given by

$$dr = d\zeta_g. \quad (2.3.70)$$

Note further that the derivative of s with respect to ζ_g was previously given by (2.3.45). After a substitution from (2.3.68) into (2.3.45), it is known that

$$\frac{ds}{d\zeta_g} = \zeta_g (2\pi)^{-.5} \exp(-.5\, \zeta_g^2). \quad (2.3.71)$$

Therefore, the differential of s with respect to that of ζ_g is provided by

$$ds = \zeta_g (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g. \quad (2.3.72)$$

Continuing with integration by parts, the simple differential of a product is used:

$$d(r\, s) = r\, ds + s\, dr. \quad (2.3.73)$$

Transposing (2.3.73), it is found that

$$r\, ds = d(r\, s) - s\, dr \quad (2.3.74)$$

where substitutions from (2.3.67), (2.3.68), (2.3.70) and (2.3.72) into (2.3.74) are now possible. These substitutions result in

$$\zeta_g^2 (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g = d[-\zeta_g(2\pi)^{-.5} \exp(-.5\, \zeta_g^2)] + (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g. \quad (2.3.75)$$

Upon integrating (2.3.75), it is known that

$$\int_{\gamma_g}^{\infty} \zeta_g^2 (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g$$

$$= \int_{\gamma_g}^{\infty} d[-\zeta_g(2\pi)^{-.5} \exp(-.5\, \zeta_g^2)] + \int_{\gamma_g}^{\infty} (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g, \quad (2.3.76)$$

where the first integral on the right side of the equality requires further attention. Because of (2.3.68), it is known that

$$\int_{\gamma_g}^{\infty} d[-\zeta_g(2\pi)^{-.5} \exp(-.5\, \zeta_g^2)] = \int_{\gamma_g}^{\infty} d[-\zeta_g\, \phi(\zeta_g)]. \quad (2.3.77)$$

Since the integrand in (2.3.77) is unity, the antiderivative is merely the variable of integration, $-\zeta_g\phi(\zeta_g)$. A solution for this integral is obtained by evaluating the limit of the integral as the upper limit of integration, b, approaches positive infinity. It is known, therefore, that

$$\int_{\gamma_g}^{\infty} d[-\zeta_g(2\pi)^{-.5}\exp(-.5\,\zeta_g^2)] = \lim_{b\to\infty}\int_{\gamma_g}^{b} d[-\,\zeta_g\phi(\zeta_g)] = \lim_{b\to\infty}\,[-b\,\phi(b) + \gamma_g\,\phi(\gamma_g)], \qquad (2.3.78)$$

where the antiderivative evaluated at b, $-b\,\phi(b)$, approaches zero as b becomes positively infinite. It is then found that

$$\int_{\gamma_g}^{\infty} d[-\,\zeta_g(2\pi)^{-.5}\exp(-.5\,\zeta_g^2) = \gamma_g\,\phi(\gamma_g). \qquad (2.3.79)$$

Substitutions from (2.3.79) and (2.3.31) into (2.3.76) allow one to write

$$\int_{\gamma_g}^{\infty} \zeta_g^2\,(2\pi)^{-.5}\exp(-.5\,\zeta_g^2)\,d\,\zeta_g = \gamma_g\,\phi(\gamma_g) + P_g \qquad (2.3.80)$$

which can now be substituted into (2.3.66) providing

$$\mathscr{E}\,(Z_g^2 \mid u_g = 1) = 1 + \frac{\gamma_g\,\phi(\gamma_g)}{P_g}. \qquad (2.3.81)$$

Substitutions from (2.3.81) and from the squared result of (2.3.48) into (2.3.60), along with some rearrangement, now yield the desired result

$$\sigma^2\,(Z_g \mid u_g = 1) = 1 - \frac{\phi(\gamma_g)}{P_g}\left(\frac{\phi(\gamma_g)}{P_g} - \gamma_g\right) \qquad (2.3.82)$$

as a convenient expression for the variance of Z_g given a correct answer to item g.

The Variance of Z_g Given an Incorrect Answer to Item g, $\sigma^2(Z_g \mid u_g = 0)$. In solving for the variance of Z_g given an incorrect answer to item g, an expression is found for $\mathscr{E}(Z_g^2 \mid u_g = 0)$. This quantity represents the expected value for the sum of squared deviations from the mean that was appropriate before the truncation of Z_g at γ_g. Later, an expression is obtained for the desired variance or the sum of squared deviations about the mean subsequent to this truncation, $\sigma^2(Z_g \mid u_g = 0)$, through

$$\sigma^2\,(Z_g \mid u_g = 0) = \mathscr{E}(Z_g^2 \mid u_g = 0) - \mu^2(Z_g \mid u_g = 0), \qquad (2.3.83)$$

which is an explicit solution for $\sigma^2(Z_g \mid u_g = 0)$ given the identity

$$\mathscr{E}(Z_g^2 \mid u_g = 0) = \sigma^2(Z_g \mid u_g = 0) + \mu^2(Z_g \mid u_g = 0). \qquad (2.3.84)$$

Proceeding with the solution for $\mathscr{E}(Z_g^2 \mid u_g = 0)$, it is known by definition that

$$\mathscr{E}(Z_g^2 \mid u_g = 0) = \int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \zeta_g^2\,\phi^{\dagger}(\zeta_g,\theta)\,d\theta\,d\zeta_g, \qquad (2.3.85)$$

where $\phi^{\dagger}(\zeta_g,\theta)$ is provided in (2.3.50) and verified as the joint density function in the discussion surrounding (2.3.52). A substitution from (2.3.50) into (2.3.85) allows one to write

$$\mathscr{E}(Z_g^2 \mid u_g = 0) = \frac{1}{Q_g}\int_{\infty}^{\gamma_g}\int_{-\infty}^{\infty} \zeta_g^2\,\phi(\zeta_g,\theta)\,d\theta\,d\zeta_g. \qquad (2.3.86)$$

After substituting from (2.3.16) into (2.3.86) and continuing with the integration, one may write

$$\mathscr{E}(Z_g^2 \mid u_g = 0)$$
$$= \frac{1}{Q_g}\int_{-\infty}^{\gamma_g} \zeta_g^2\,(2\pi)^{-.5}\exp(-.5\zeta_g^2)\,d\zeta_g \int_{-\infty}^{\infty}(2\pi)^{-.5}\,[1 - \rho^2(Z_g,\Theta)]^{-.5}\exp\left(-.5\left\{\frac{\theta - \rho(Z_g,\Theta)\,\zeta_g}{[1 - \rho^2(Z_g,\Theta)]^{.5}}\right\}^2\right)d\theta, \qquad (2.3.87)$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\mathscr{E}(Z_g^2 \mid u_g = 0) = \frac{1}{Q_g}\int_{-\infty}^{\gamma_g} \zeta_g^2\,(2\pi)^{-.5}\exp(-.5\,\zeta_g^2)\,d\zeta_g \int_{-\infty}^{\infty}(2\pi)^{-.5}\exp\{-.5\,[\bar{\theta}(\zeta_g)]^2\}\,d\bar{\theta}(\zeta_g). \qquad (2.3.88)$$

Again, it is to be noted that the rightmost integral is unity, which allows one to write

$$\mathscr{E}(Z_g^2 \mid u_g = 0) = \frac{1}{Q_g} \int_{-\infty}^{\gamma_g} \zeta_g^2 \, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g, \tag{2.3.89}$$

where it is customary to solve this integral through the method of integration by parts. Again, the definitions provided in (2.3.67) and (2.3.68) along with their consequences as given in (2.3.70) and (2.3.72) are used. Continuing with integration by parts, the simple differential of a product is again used as in (2.3.73) and (2.3.74), where substitution from (2.3.67), (2.3.68), (2.3.70), and (2.3.72) into (2.3.74) again yields (2.3.75). Upon integrating (2.3.75) between the different limits, one now obtains

$$\begin{aligned}\int_{-\infty}^{\gamma_g} \zeta_g^2 \, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g \\ = \int_{-\infty}^{\gamma_g} d[-\zeta_g \, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)] + \int_{-\infty}^{\gamma_g} (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g,\end{aligned} \tag{2.3.90}$$

where the first integral on the right side of the equality again requires further attention. Because of (2.3.68), it is known that

$$\int_{-\infty}^{\gamma_g} d[-\zeta_g \, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)] = \int_{-\infty}^{\gamma_g} d[-\zeta_g \, \phi(\zeta_g)]. \tag{2.3.91}$$

Since the integrand in (2.3.91) is unity, the antiderivative is merely the variable of integration, $-\zeta_g \, \phi(\zeta_g)$. A solution for the integral in (2.3.91) is obtained by evaluating the limit of the integral as the lower limit of integration, a, approaches negative infinity. One therefore has

$$\begin{aligned}\int_{-\infty}^{\gamma_g} d[-\zeta_g \, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)] &= \lim_{a \to -\infty} \int_{a}^{\gamma_g} d[-\zeta_g \, \phi(\zeta_g)] \\ &= \lim_{a \to -\infty} [-\gamma_g \, \phi(\gamma_g) + a \, \phi(a)],\end{aligned} \tag{2.3.92}$$

where the antiderivative evaluated at a, $-a \, \phi(a)$, approaches zero as a becomes negatively infinite. The solution for this integral, then, is

$$\int_{-\infty}^{\gamma_g} d[-\zeta_g \, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)] = -\, \gamma_g \, \phi(\gamma_g). \tag{2.3.93}$$

Substitutions from (2.3.93) and (2.3.35) into (2.3.90) allow one to write

$$\int_{-\infty}^{\gamma_g} \zeta_g^2 \, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g = -\, \gamma_g \, \phi(\gamma_g) + Q_g, \tag{2.3.94}$$

which can now be substituted into (2.3.89), yielding

$$\mathscr{E}(Z_g^2 \mid u_g = 0) = 1 - \frac{\gamma_g \, \phi(\gamma_g)}{Q_g}. \tag{2.3.95}$$

Substitutions from (2.3.95) and from the squared result of (2.3.59) into (2.3.83), along with some rearrangement, now yield the desired result

$$\sigma^2(Z_g \mid u_g = 0) = 1 - \frac{\phi(\gamma_g)}{Q_g} \left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g \right] \tag{2.3.96}$$

as a convenient expression for the variance of Z_g given an incorrect answer to item g.

The expressions for the variances provided in (2.3.82) and (2.3.96) will have further use in later derivations in Chapter 5. Specifically, these equations will provide input for the derivations of the restricted variances of ability given the realizations of the binary variable U_g.

The Least Squares Estimators of Ability Given the Realizations of U_g

Since the mean is the point about which the sum of squared discrepancies is minimized, least squares estimators of ability for the possible realizations of the binary variable U_g are readily obtained.

Specific Case. For the specific case where Θ has a mean of zero and a variance of unity these estimators are merely the means of Θ given the two possible outcomes: the mean of Θ given a correct answer to item g, $\mu(\Theta \mid u_g = 1)$, and the mean of Θ given an incorrect answer to item g, $\mu(\Theta \mid u_g = 0)$. These least squares estimators of ability will now be derived for the specific case.

Specific Case: The Mean of Θ Given a Correct Answer to Item g, $\mu(\Theta \mid u_g = 1)$. In order to obtain a convenient expression for the mean of Θ given a correct answer to item g, $\mu(\Theta \mid u_g = 1)$, it is necessary to evaluate the defining relationship

$$\mu(\Theta \mid u_g = 1) = \mathscr{E}(\Theta \mid u_g = 1) = \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta\, \phi^*(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{2.3.97}$$

where $\phi^*(\zeta_g,\theta)$ is given in (2.3.37) and verified as the joint density function in the discussion surrounding (2.3.39). A substitution from (2.3.37) into (2.3.97) allows one to write

$$\mu(\Theta \mid u_g = 1) = \mathscr{E}(\Theta \mid u_g = 1) = \frac{1}{P_g} \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g. \tag{2.3.98}$$

After a substitution from (2.3.16) into (2.3.98), one may write

$$\mu(\Theta \mid u_g = 1) = \mathscr{E}(\Theta \mid u_g = 1)$$

$$= \frac{1}{P_g} \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta\, (2\pi)^{-.5}\, [1 - \rho^2(Z_g,\Theta)]^{-.5} \exp\left(-.5 \left\{ \frac{\theta - \rho(Z_g,\Theta)\, \zeta_g}{[1 - \rho^2(Z_g,\Theta)]^{.5}} \right\}^2 \right) d\theta\, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g, \tag{2.3.99}$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\mu(\Theta \mid u_g = 1) = \mathscr{E}(\Theta \mid u_g = 1)$$
$$= \frac{1}{P_g} \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta\, (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g)\, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g. \tag{2.3.100}$$

The innermost integral in (2.3.100) is merely the expected value of Θ given ζ_g, $\mathscr{E}(\Theta \mid \zeta_g)$, which can be represented in Figure 2.3.1 as the value of the coordinate of Θ for a point on the line of regression of Θ onto ζ_g labelled $\mathscr{E}(\Theta \mid \zeta_g)$, where the coordinate of Z_g is the particular fixed value ζ_g. The value that this integral assumes is merely the product $\rho(Z_g,\Theta)\, \zeta_g$, as will soon become evident. Proceeding with an evaluation of this innermost integral, one solves (2.3.18) explicitly for θ where this solution is

$$\theta = [1 - \rho^2(Z_g,\Theta)]^{.5}\, \tilde{\theta}(\zeta_g) + \rho(Z_g,\Theta)\, \zeta_g. \tag{2.3.101}$$

After a substitution of (2.3.101) into the innermost integral in (2.3.100), one can write

$$\int_{-\infty}^{\infty} \theta\, (2\pi)^{-.5} \exp\{-.5\, [\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g)$$

$$= \int_{-\infty}^{\infty} \{[1 - \rho^2(Z_g,\Theta)]^{.5}\, \tilde{\theta}(\zeta_g) + \rho(Z_g,\Theta)\, \zeta_g\}\, (2\pi)^{-.5} \exp\{-.5\, [\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g). \tag{2.3.102}$$

The integral on the right side of (2.3.102) can be expressed as a sum of two integrals

$$\int_{-\infty}^{\infty} \theta\, (2\pi)^{-.5} \exp\{-.5\, [\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g)$$

$$\begin{aligned} &= [1 - \rho^2(Z_g,\Theta)]^{.5} \int_{-\infty}^{\infty} \tilde{\theta}(\zeta_g)\, (2\pi)^{-.5} \exp\{-.5\, [\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g) \\ &\quad + \rho(Z_g,\Theta)\, \zeta_g \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5\, [\tilde{\theta}(\zeta_g)]^2\}\, d\tilde{\theta}(\zeta_g), \end{aligned} \tag{2.3.103}$$

where some of the constants in the integration have been written outside of the integrals for convenience. The first integral on the right side of the equality in (2.3.103) is merely the mean of $\Theta(\zeta_g)$, $\mu[\Theta(\zeta_g)]$ which is known to be zero. Thus, the product represented by the first term on the right side of the equality in (2.3.103) is also zero. The second integral on the right side of the equality represents the total area in the standard normal distribution which is known to be unity. Thus, one may write

$$\int_{-\infty}^{\infty} \theta\, (2\pi)^{-.5} \exp\{-.5\, [\bar{\theta}(\zeta_g)]^2\}\, d\bar{\theta}(\zeta_g) = \rho(Z_g,\Theta)\, \zeta_g, \tag{2.3.104}$$

which can be substituted into (2.3.100), producing

$$\begin{aligned}\mu(\Theta \mid u_g = 1) &= \mathscr{E}(\Theta \mid u_g = 1)\\ &= \frac{1}{P_g}\int_{\gamma_g}^{\infty} \rho(Z_g,\Theta)\, \zeta_g\, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g.\end{aligned} \tag{2.3.105}$$

Since $\rho(Z_g,\Theta)$ is a constant in the integration of (2.3.105), one has

$$\begin{aligned}\mu(\Theta \mid u_g = 1) &= \mathscr{E}(\Theta \mid u_g = 1)\\ &= \rho(Z_g,\Theta)\, \frac{1}{P_g}\int_{\gamma_g}^{\infty} \zeta_g\, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g,\end{aligned} \tag{2.3.106}$$

where a substitution from (2.3.43) yields

$$\mu(\Theta \mid u_g = 1) = \rho(Z_g,\Theta)\, \mu(Z_g \mid u_g = 1) \tag{2.3.107}$$

which may also be expressed as

$$\mu(\Theta \mid u_g = 1) = \rho(Z_g,\Theta)\, \frac{\phi(\gamma_g)}{P_g} \tag{2.3.108}$$

because of (2.3.48). Equation (2.3.108) represents a convenient expression for the least squares estimator of Θ given a correct answer to item g. This estimator is merely the product of a regression weight—in this case the correlation $\rho(Z_g,\Theta)$, because both Z_g and Θ have a standard deviation of unity—and the value of the density at γ_g in the standard normal distribution, $\phi(\gamma_g)$, divided by the unconditional probability of a correct answer to item g, P_g.

Specific Case: The Mean of Θ Given an Incorrect Answer to Item g, $\mu(\Theta \mid u_g = 0)$. In order to obtain an expression for the mean of Θ given an incorrect answer to item g one must evaluate the defining relationship

$$\mu(\Theta \mid u_g = 0) = \mathscr{E}(\Theta \mid u_g = 0) = \int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \theta\, \phi^{\dagger}(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{2.3.109}$$

where $\phi^{\dagger}(\zeta_g,\theta)$ is given in (2.3.50) and verified as the joint density function in the discussion surrounding (2.3.52). A substitution from (2.3.50) into (2.3.109) allows one to write

$$\mu(\Theta \mid u_g = 0) = \mathscr{E}(\Theta \mid u_g = 0) = \frac{1}{Q_g}\int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \theta\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g. \tag{2.3.110}$$

After substitution from (2.3.16) into (2.3.110), one may write

$$\begin{aligned}&\mu(\Theta \mid u_g = 0) = \mathscr{E}(\Theta \mid u_g = 0)\\ &= \frac{1}{Q_g}\int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \theta\, (2\pi)^{-.5}\, [1 - \rho^2(Z_g,\Theta)]^{-.5} \exp\left(-.5\left\{\frac{\theta - \rho(Z_g,\Theta)\, \zeta_g}{[1 - \rho^2(Z_g,\Theta)]^{.5}}\right\}^2\right) d\theta\, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g,\end{aligned} \tag{2.3.111}$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\begin{aligned}\mu(\Theta \mid u_g = 0) &= \mathscr{E}(\Theta \mid u_g = 0)\\ &= \frac{1}{Q_g}\int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \theta\, (2\pi)^{-.5} \exp\{-.5\, [\bar{\theta}(\zeta_g)]^2\}\, d\bar{\theta}(\zeta_g)\, (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g.\end{aligned} \tag{2.3.112}$$

The innermost integral in (2.3.112) is again equal to $\mathscr{E}(\Theta \mid \zeta_g)$, or the product $\rho(Z_g,\Theta)\ \zeta_g$ for each particular value of Z_g as proved in (2.3.104). A substitution from (2.3.104) into (2.3.112) then allows the writing of

$$\begin{aligned}\mu(\Theta \mid u_g = 0) &= \mathscr{E}(\Theta \mid u_g = 0)\\ &= \frac{1}{Q_g}\int_{-\infty}^{\gamma_g} \rho(Z_g,\Theta)\ \zeta_g (2\pi)^{-.5} \exp(-.5\ \zeta_g^2)\ d\zeta_g,\end{aligned} \tag{2.3.113}$$

where the constant $\rho(Z_g,\Theta)$ may be written outside of the integral. One then has

$$\begin{aligned}\mu(\Theta \mid u_g = 0) &= \mathscr{E}(\Theta \mid u_g = 0)\\ &= \rho(Z_g,\Theta)\ \frac{1}{Q_g}\int_{-\infty}^{\gamma_g} \zeta_g (2\pi)^{-.5} \exp(-.5\ \zeta_g^2)\ d\zeta_g,\end{aligned} \tag{2.3.114}$$

where a substitution from (2.3.56) allows one to write

$$\mu(\Theta \mid u_g = 0) = \rho(Z_g,\Theta)\ \mu(Z_g \mid u_g = 0) \tag{2.3.115}$$

which may also be expressed as

$$\mu(\Theta \mid u_g = 0) = -\ \rho(Z_g,\Theta)\ \frac{\phi(\gamma_g)}{Q_g} \tag{2.3.116}$$

because of (2.3.59). Equation (2.3.116) provides a convenient expression for the least squares estimator of Θ given an incorrect answer to item g. This estimator is merely the negative of the product of a regression weight—in this case, $\rho(Z_g,\Theta)$, because both Z_g and Θ have standard deviations of unity—and the value of the density at γ_g in the standard normal distribution, $\phi(\gamma_g)$, divided by the unconditional probability of an incorrect answer to item g, Q_g.

General Case. The general case allows the examiner to prescribe the scale of ability or the mean and variance of Θ^*, $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$. Under this case, Θ may be viewed as resulting from the standardization of Θ^*

$$\theta = \frac{\theta^* - \mu(\Theta^*)}{\sigma(\Theta^*)} \tag{2.3.117}$$

where this relationship indicates that θ and θ^* are equal when $\mu(\Theta^*)$ equals zero and $\sigma(\Theta^*)$ equals unity. Thus, the general case subsumes the specific case. In the general case, it is to be noted that

$$\theta^* = \theta\ \sigma(\Theta^*) + \mu(\Theta^*) \tag{2.3.118}$$

when (2.3.117) is solved explicitly for θ^*.

A note of explanation is in order for applications of the general case. The item parameters, a_g and b_g of (2.1.12) and (2.1.13), respectively, are not appropriate, except in one instance, for the general case. The sole exception is the subsumed specific case. For the general case, these parameters require transformations that produce a particular invariant result. When a desired mean, $\mu(\Theta^*)$, and variance, $\sigma^2(\Theta^*)$, have been prescribed, the particular transformations are then known. These transformations provide

$$a_g^* = \frac{a_g}{\sigma(\Theta^*)} \tag{2.3.119}$$

and

$$b_g^* = b_g\ \sigma(\Theta^*) + \mu(\Theta^*) \tag{2.3.120}$$

that are appropriate in the general case which, of course, subsumes the specific case.

The point of dichotomization on the standardized conditional distribution of Z_g given θ^* may be defined as

$$\gamma_g(\theta^*) = -\ a_g^*\ (\theta^* - b_g^*). \tag{2.3.121}$$

The particular invariant result which must be produced by the transformations in (2.3.119) and (2.3.120) then requires that

$$\gamma_g(\theta) = \gamma_g(\theta^*) \tag{2.3.122}$$

which maintains an identical point of dichotomization on the $\tilde{Z}_g(\theta)$ and the $\tilde{Z}_g(\theta^*)$ under a change in the scale of ability. This result provides that the probabilities yielding the item characteristic curve remain undisturbed by this transformation of the scale of ability. Substitutions from (2.1.15) and (2.3.121) into (2.3.122) yield

$$-a_g\ (\theta - b_g) = -a_g^*\ (\theta^* - b_g^*). \tag{2.3.123}$$

Subsequent substitutions from (2.3.118), (2.3.119), and (2.3.120) into (2.3.123) provide the obvious equality

$$-a_g\ (\theta - b_g) = -a_g\ (\theta - b_g) \tag{2.3.124}$$

in proof of the required invariant result and the appropriateness of (2.3.119) and (2.3.120) for the general case. Notice that the property of invariance does not pertain to the parameters, but to the result produced by particular transformations of the parameters. This relationship is made explicit in (2.3.123).

The property of invariance illustrated for (2.3.122) produces a desirable result. This result guarantees that the probability of obtaining a correct answer on item g remains invariant under arbitrary prescriptions for the scale of ability, that is the mean and variance of Θ^*, $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$. Later in the discussion, there will be further cause to consider the property of invariance.

For subsequent developments, expressions are required for the correlation, $\rho(Z_g,\Theta)$, and the point of dichotomization on the continuous variable Z_g, γ_g, given the parameters a_g^* and b_g^*. These expressions will be derived here for future reference.

In order to obtain an expression for $\rho(Z_g,\Theta)$ given a_g^*, (2.3.119) is solved explicitly for a_g. This solution yields

$$a_g = a_g^*\ \sigma(\Theta^*) \tag{2.3.125}$$

which can then be substituted into (2.3.28) to provide the desired results. After substitutions from (2.3.125) into (2.3.28), one has

$$\rho(Z_g,\Theta) = \frac{a_g^*\ \sigma(\Theta^*)}{\{1 + [a_g^*\ \sigma(\Theta^*)]^2\}^{.5}}, \tag{2.3.126}$$

an expression appropriate for the general case.

An expression for γ_g given b_g^* can be obtained through solving (2.3.120) explicitly for b_g. This solution yields

$$b_g = \frac{b_g^* - \mu(\Theta^*)}{\sigma(\Theta^*)} \tag{2.3.127}$$

which can then be substituted along with (2.3.126) into (2.3.27) to provide the desired result. After the substitution of (2.3.126) and (2.3.127) into (2.3.27), one may write

$$\gamma_g = \frac{b_g^* - \mu(\Theta^*)}{[(a_g^*)^{-2} + \sigma^2(\Theta^*)]^{.5}}, \tag{2.3.128}$$

an expression appropriate for the general case.

For the general case, the least squares estimators of ability are merely the means of Θ^* given the two possible outcomes: the mean of Θ^* given a correct answer to item g, $\mu(\Theta^* \mid u_g = 1)$, and the mean of Θ^* given an incorrect answer to item g, $\mu(\Theta^* \mid u_g = 0)$. These estimators of ability will now be derived for this general case.

General Case: The Mean of Θ^* Given a Correct Answer to Item g, $\mu(\Theta^* \mid u_g = 1)$. In order to obtain an expression for the mean of Θ^* given a correct answer to item g, one must evaluate the defining relationship

$$\mu(\Theta^* \mid u_g = 1) = \mathscr{E}(\Theta^* \mid u_g = 1) = \int_{\gamma_g}^{\infty}\int_{-\infty}^{\infty} \theta^*\ \phi^*(\zeta_g,\theta)\ d\theta\ d\zeta_g, \tag{2.3.129}$$

where $\phi^*(\zeta_g,\theta)$ is given in (2.3.37), verified as the joint density function in the discussion surrounding (2.3.39), and still appropriate because of (2.3.117) and (2.3.118). A substitution from (2.3.37) into (2.3.129) allows one to write

$$\mu(\Theta^* \mid u_g = 1) = \mathscr{E}(\Theta^* \mid u_g = 1) = \frac{1}{P_g}\int_{\gamma_g}^{\infty}\int_{-\infty}^{\infty} \theta^*\ \phi(\zeta_g,\theta)\ d\theta\ d\zeta_g. \tag{2.3.130}$$

After a substitution from (2.3.16) into (2.3.130), one may write

$$\mu(\Theta^* \mid u_g = 1) = \mathscr{E}(\Theta^* \mid u_g = 1)$$
$$= \frac{1}{P_g} \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta^* (2\pi)^{-.5} [1 - \rho^2(Z_g,\Theta)]^{-.5} \exp\left(-.5 \left\{ \frac{\theta - \rho(Z_g,\Theta)\ \zeta_g}{[1 - \rho^2(Z_g,\Theta)]^{.5}} \right\}^2 \right) d\theta\, (2\pi)^{-.5} \exp(-.5\ \zeta_g^2)\ d\zeta_g, \tag{2.3.131}$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\mu(\Theta^* \mid u_g = 1) = \mathscr{E}(\Theta^* \mid u_g = 1)$$
$$= \frac{1}{P_g} \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta^*\ (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g)\ (2\pi)^{-.5} \exp(-.5\ \zeta_g^2)\ d\zeta_g. \tag{2.3.132}$$

The innermost integral in (2.3.132) is merely the expected value of Θ^* given ζ_g, $\mathscr{E}(\Theta^* \mid \zeta_g)$, which represents the value of the coordinate of Θ^* for a point on the line of regression of Θ^* onto Z_g for the particular fixed value ζ_g. Proceeding with an evaluation of (2.3.132), a substitution from (2.3.118) into the innermost integral in (2.3.132) yields

$$\int_{-\infty}^{\infty} \theta^*\ (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g)$$
$$= \int_{-\infty}^{\infty} [\theta\ \sigma(\Theta^*) + \mu(\Theta^*)]\ (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g). \tag{2.3.133}$$

The integral on the right side of (2.3.133) can be expressed as a sum of two integrals

$$\int_{-\infty}^{\infty} \theta^*\ (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g)$$
$$= \sigma(\Theta^*) \int_{-\infty}^{\infty} \theta\ (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g)$$
$$+ \mu(\Theta^*) \int_{-\infty}^{\infty} (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g), \tag{2.3.134}$$

where some of the constants in the integration have been written outside of the integrals for convenience. The first integral on the right side of the equality in (2.3.134) is merely the product $\rho(Z_g,\Theta)\ \zeta_g$ as given by (2.3.104). Thus, the first term on the right side of the equality in (2.3.134) is merely the triple product $\rho(Z_g,\Theta)\ \sigma(\theta^*)\ \zeta_g$. The second integral on the right side of the equality in (2.3.134) represents the total area in the standard normal distribution which is known to be unity. Thus, the second term on the right side of the equality in (2.3.134) is merely $\mu(\Theta^*)$. Therefore, one may write

$$\int_{-\infty}^{\infty} \theta^*\ (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g) = \mu(\Theta^*) + \rho(Z_g,\Theta)\ \sigma(\Theta^*)\ \zeta_g, \tag{2.3.135}$$

which can be substituted into (2.3.132) producing

$$\mu(\Theta^* \mid u_g = 1) = \mathscr{E}(\Theta^* \mid u_g = 1)$$
$$= \frac{1}{P_g} \int_{\gamma_g}^{\infty} [\mu(\Theta^*) + \rho(Z_g,\Theta)\ \sigma(\Theta^*)\ \zeta_g]\ (2\pi)^{-.5} \exp(-.5\zeta_g^2)\ d\zeta_g, \tag{2.3.136}$$

where the rightmost equality in (2.3.136) can now be expressed as the sum of two integrals. One may write

$$\mu(\Theta^* \mid u_g = 1) = \mathscr{E}(\Theta^* \mid u_g = 1)$$
$$= \frac{\mu(\Theta^*)}{P_g} \int_{\gamma_g}^{\infty} (2\pi)^{-.5} \exp(-.5\ \zeta_g^2)\ d\zeta_g$$
$$+ \rho(Z_g,\Theta)\ \frac{\sigma(\Theta^*)}{P_g} \int_{\gamma_g}^{\infty} \zeta_g\ (2\pi)^{-.5} \exp(-.5\ \zeta_g^2)\ d\zeta_g, \tag{2.3.137}$$

where some of the constants in the integration have been written outside of the integrals for convenience. Substitutions from (2.3.31) and (2.3.43) into (2.3.137) yield

$$\mu(\Theta^* \mid u_g = 1) = \mu(\Theta^*) + \rho(Z_g,\Theta)\ \sigma(\Theta^*)\ \mu(Z_g \mid u_g = 1); \tag{2.3.138}$$

which may also be expressed as

$$\mu(\Theta^* \mid u_g = 1) = \mu(\Theta^*) + \rho(Z_g,\Theta)\ \sigma(\Theta^*) \frac{\phi(\gamma_g)}{P_g}, \tag{2.3.139}$$

because of (2.3.48). This estimator is merely the product of a regression weight—in this case $\rho(Z_g,\Theta)\ \sigma(\Theta^*)$ because Z_g has a standard deviation of unity—and the value of the density at γ_g in the standard normal distribution, $\phi(\gamma_g)$, divided by the unconditional probability of a correct answer to item g, P_g, added to the mean of Θ^*, $\mu(\Theta^*)$.

General Case: The Mean of Θ^* Given an Incorrect Answer to Item g, $\mu(\Theta^* \mid u_g = 0)$. In order to obtain an expression for the mean of Θ^* given an incorrect answer to item g, it is necessary to evaluate the defining relationship

$$\mu(\Theta^* \mid u_g = 0) = \mathscr{E}(\Theta^* \mid u_g = 0) = \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta^*\ \phi^\dagger(\zeta_g,\theta)\ d\theta\ d\zeta_g, \tag{2.3.140}$$

where $\phi^\dagger(\zeta_g,\theta)$ is given in (2.3.50), verified as the joint density function in the discussion surrounding (2.3.52), and still appropriate because of (2.3.117) and (2.3.118). A substitution from (2.3.50) into (2.3.140) allows one to write

$$\mu(\Theta^* \mid u_g = 0) = \mathscr{E}(\Theta^* \mid u_g = 0) = \frac{1}{Q_g} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta^*\ \phi(\zeta_g,\theta)\ d\theta\ d\zeta_g. \tag{2.3.141}$$

After a substitution from (2.3.16) into (2.3.141), it is found that

$$\mu(\Theta^* \mid u_g = 0) = \mathscr{E}(\Theta^* \mid u_g = 0)$$

$$= \frac{1}{Q_g} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta^*\ (2\pi)^{-.5}[1-\rho^2(Z_g,\Theta)]^{-.5} \exp\left(-.5\left\{ \frac{\theta-\rho(Z_g,\Theta)\zeta_g}{[1-\rho^2(Z_g,\Theta)]^{.5}} \right\}^2 \right) d\theta\ (2\pi)^{-.5} \exp(-.5\zeta_g^2)\ d\zeta_g \tag{2.3.142}$$

where substitutions from (2.3.18) and (2.3.20) provide

$$\mu(\Theta^* \mid u_g = 0) = \mathscr{E}(\Theta^* \mid u_g = 0)$$

$$= \frac{1}{Q_g} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta^*\ (2\pi)^{-.5} \exp\{-.5[\tilde{\theta}(\zeta_g)]^2\}\ d\tilde{\theta}(\zeta_g)\ (2\pi)^{-.5} \exp(-.5\zeta_g^2)\ d\zeta_g. \tag{2.3.143}$$

The innermost integral in (2.3.143) is the expected value of Θ^* given ζ_g, $\mathscr{E}(\Theta^* \mid \zeta_g)$, or merely, $\mu(\Theta^*)+\rho(Z_g,\Theta)\ \sigma(\Theta^*)\zeta_g$, as proved in (2.3.135). A substitution from (2.3.135) into (2.3.143) then allows the writing of

$$\mu(\Theta^* \mid u_g = 0) = \mathscr{E}(\Theta^* \mid u_g = 0)$$

$$= \frac{1}{Q_g} \int_{-\infty}^{\gamma_g} [\mu(\Theta^*)+\rho(Z_g,\Theta)\sigma(\Theta^*)\zeta_g]\ (2\pi)^{-.5} \exp(-.5\zeta_g^2)\ d\zeta_g, \tag{2.3.144}$$

where the rightmost equality in (2.3.144) can now be expressed as the sum of two integrals. One may write

$$\mu(\Theta^* \mid u_g = 0) = \mathscr{E}(\Theta^* \mid u_g = 0)$$

$$= \frac{\mu(\Theta^*)}{Q_g} \int_{-\infty}^{\gamma_g} (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g$$

$$+ \rho(Z_g,\Theta) \frac{\sigma(\Theta^*)}{Q_g} \int_{-\infty}^{\gamma_g} \zeta_g\, (2\pi)^{-.5} \exp(-.5\zeta_g^2)\, d\zeta_g, \tag{2.3.145}$$

where some of the constants in integration have been written outside of the integrals for convenience. Substitutions from (2.3.35) and (2.3.56) into (2.3.145) allow one to write

$$\mu(\Theta^* \mid u_g = 0) = \mu(\Theta^*) + \rho(Z_g,\Theta)\sigma(\Theta^*)\mu(Z_g \mid u_g = 0) \tag{2.3.146}$$

which may also be expressed as

$$\mu(\Theta^* \mid u_g = 0) = \mu(\Theta^*) - \rho(Z_g,\Theta)\sigma(\Theta^*) \frac{\phi(\gamma_g)}{Q_g} \tag{2.3.147}$$

because of (2.3.59). This estimator is merely the product of a regression weight—in this case $\rho(Z_g,\Theta)\sigma(\Theta^*)$ because Z_g has a standard deviation of unity—and the value of the density at γ_g in the standard normal distribution, $\phi(\gamma_g)$, divided by the unconditional probability of an incorrect answer to item g, Q_g, subtracted from the mean of $\Theta^*, \mu(\Theta^*)$.

The expressions for the least squares estimators of Θ^* provided by (2.3.138) and (2.3.139) and by (2.3.146) and (2.3.147) will have further use in later derivations in Chapter 5. Notice that the estimators provided by (2.3.108) and (2.3.116) may be viewed as resulting from (2.3.139) and (2.3.147), respectively, under the assumption that $\mu(\Theta^*)$ is zero and $\sigma^2(\Theta^*)$ is unity. Since the specific case can be directly obtained from the general case through this simplifying assumption, later developments in ability estimation will be primarily concerned with the general case. Earlier it was noted that the general case has the more interesting practical implications. In Chapter 5, the sequential estimation of Θ^* given the free-response items constituting a tailored test will become of concern.

3. EFFECTS OF BINARY SCORING: A SUBMODEL FOR MULTIPLE-CHOICE ITEMS

3.1 The Three-Parameter Normal Ogive Submodel

Consider the situation where the possible answers to a question posed by an item have been assigned ordered scores on the basis of their relative correctness. Suppose that it is then possible to assign recognition scores to individuals. Let this score for a given individual be defined as the value of the most correct answer that would be recognized as correct on the given occasion. For p items designed to measure the same ability, the ordering of scores for the possible answers generates p continua of item scores. When the frequencies of individuals with identical recognition scores are associated with their corresponding ordered scores, random variables such as the Z_g of Chapter 1 are the result.

In the case of a multiple-choice item, one of its alternatives serves as a correct answer. The score assigned to the answer chosen for the correct alternative is then γ_g, the point of dichotomization on the random variable Z_g. As a result, individuals with recognition scores equal to or greater than γ_g, that is those who recognize the correct alternative as well as better answers than the correct alternative as correct, receive a binary score of *one*. The remaining individuals then guess. Successful guessing on the given item occurs at a specific rate, and this rate of success occurs at random among those individuals who are unable to recognize the correct alternative on the given occasion. Individuals who guess successfully also receive a binary score of *one*, while those who do not receive a binary score of *zero*. Hence, U'_g has two possible realizations:

$u'_g = 1$ When $\zeta_g \geqslant \gamma_g$ (correct alternative recognized) or
$\zeta_g < \gamma_g$ (correct alternative not recognized: guessing successful)
$u'_g = 0$ When $\zeta_g < \gamma_g$ (correct alternative not recognized: guessing unsuccessful)

where the prime on U'_g indicates that item g, a multiple-choice item, is influenced by guessing. Most of the expressions in this section will have their counterparts in the two-parameter normal ogive submodel. As in the instance of u'_g, some of these expressions will be directly influenced by guessing. To maintain clarity in the discussion, the expressions influenced by guessing will be denoted through the use of a prime. Unprimed expressions will be those uninfluenced by guessing that are identically defined under both submodels.

The rate of successful guessing on item g is not solely determined by the number of its alternatives. For instance, the relative attractiveness of the alternatives will have some influence. An alternative can appear more or less attractive than others to those unable to recognize the correct answer. The condition is permissible because this submodel does not assume that guessing occurs at random with respect to the item alternatives when the correct alternative is not recognized. The requirement that individuals respond at random to the item alternatives when the correct alternative is not recognized goes beyond the statement of this submodel. The submodel merely posits that obtaining a correct answer to item g through guessing occurs at random at a specifiable rate c_g for the subpopulation on the subinterval of Z_g where ζ_g is less than γ_g.

The proposition that extant multiple-choice items contain alternatives of equal attractiveness—a necessary condition to justify the assumption that guessing occurs at random with respect to the item alternatives—is readily disconfirmed with empirical data.

In conventional item analyses, one finds that the proportions responding to the incorrect alternatives of each multiple-choice item g are not always—as required by this proposition—equal. The typical observation is that of the inequality of these proportions.

Let c_g represent the rate of successful guessing on item g when ζ_g is less than γ_g. As a consequence, c_g, as will be seen in (3.1.17), is defined as, or fixed by, the lower asymptote of the item characteristic curve. Since c_g is fixed, Z_g then completely determines the binary random variable U'_g. Again the Z_g are affected by measurement error. Thus the U'_g are likewise affected. The transformation of Z_g into U'_g again results in a transformation of the true score metric. A new true score variable T'_g and a new error score variable E'_g are again implicitly produced by the transformation of Z_g into U'_g. For any individual i on occasion o, t'_{g_i} represents the individual's true score on binary item g while e'_{gio} represents the individual's error score on binary item g. True score is defined as the expected value of the u'_{gio} for individual i over testing occasions, that is,

$$t'_{gi} = \mathscr{E}_o\, U'_{gio} \tag{3.1.1}$$

and error score is defined as

$$e'_{gio} = u'_{gio} - t'_{gi}. \tag{3.1.2}$$

Given the definition of true score presented in (3.1.1), T'_g is the least squares estimate of U'_g given θ. All individuals with ability equal to that of individual $i, \theta = \theta_i$, have identical true scores,

$$t'_g = \mathscr{E}(U'_g \mid \theta = \theta_i), \tag{3.1.3}$$

which is a realization of the random variable

$$T'_g = \mathscr{E}(U'_g \mid \theta). \tag{3.1.4}$$

The expectation relationship in (3.1.4) defines T'_g as the least squares regression function of U'_g onto Θ.

For a multiple-choice item g, the functional form of the regression of U'_g onto Θ is the item characteristic curve for the three-parameter normal ogive of latent trait theory (Lord & Novick, 1968). In this case, the expected value of U'_g given θ *is*

$$\begin{aligned} T'_g = \mathscr{E}(U'_g \mid \theta) &= \sum_{u'_{gk}=0}^{1} u'_{gk} \Pr(u'_{gk} \mid \theta) \\ &= (0) \Pr(u'_g = 0 \mid \theta) + (1) \Pr(u'_g = 1 \mid \theta) \\ &= \Pr(u'_g = 1 \mid \theta) \end{aligned} \tag{3.1.5}$$

where the probability of a correct answer given θ has changed in relation to the two-parameter case. Using the definitions of $\tilde{\zeta}_g(\theta)$ and $\gamma_g(\theta)$ of (2.1.10) and (2.1.11), it is found that

$$\Pr(u'_g = 1 \mid \theta) = \Pr[\tilde{Z}_g(\theta) \geqslant \gamma_g(\theta)] + c_g \Pr[\tilde{Z}_g(\theta) < \gamma_g(\theta)] \tag{3.1.6}$$

which flows from the discussion of the manner in which Z_g determines U'_g. The hypothetical relation between Z_g and U'_g is portrayed in Figure 3.1.1, where the shaded areas on each conditional distribution, $\tilde{Z}_g(\theta)$, represent the terms in the rightmost equality of (3.1.6). Since the area under the standard normal distribution is unity, it is to be noted that due to (2.1.16)

$$\Pr[\tilde{Z}_g(\theta) < \gamma_g(\theta)] = 1 - \Pr[\tilde{Z}_g(\theta) \geqslant \gamma_g(\theta)], \tag{3.1.7}$$

which allows some simplification in (3.1.6). The substitution of (3.1.7) into (3.1.6) yields

$$\Pr(u'_g = 1 \mid \theta) = \Pr[\tilde{Z}_g(\theta) \geqslant \gamma_g(\theta)] + c_g \{1 - \Pr[\tilde{Z}_g(\theta) \geqslant \gamma_g(\theta)]\} \tag{3.1.8}$$

or, after consulting the identities provided by (2.1.16) and (2.1.20) leads to

$$\Pr(u'_g = 1 \mid \theta) = P_g(\theta) + c_g [1 - P_g(\theta)] = P_g(\theta) + c_g Q_g(\theta) \tag{3.1.9}$$

which can be compared to its two-parameter counterpart for free-response items. This expected value is designated as $P'_g(\theta)$ in order to distinguish it from its corresponding term in the two-parameter submodel and acknowledge the contribution of guessing as indexed by the rightmost terms in each of the two equalities on the right side of (3.1.9). This designation yields

$$P'_g(\theta) = P_g(\theta) + c_g [1 - P_g(\theta)] = P_g(\theta) + c_g Q_g(\theta) \tag{3.1.10}$$

which indicates that the probability of obtaining a correct answer on item g is equal to the probability of recognizing the correct alternative, $P_g(\theta)$, plus guessing correctly at a rate, c_g, when with probability, $[1 - P_g(\theta)]$ or $Q_g(\theta)$, the correct answer is not recognized. For computational convenience, (3.1.10) is usually rearranged as

$$P'_g(\theta) = c_g + (1 - c_g) P_g(\theta). \tag{3.1.11}$$

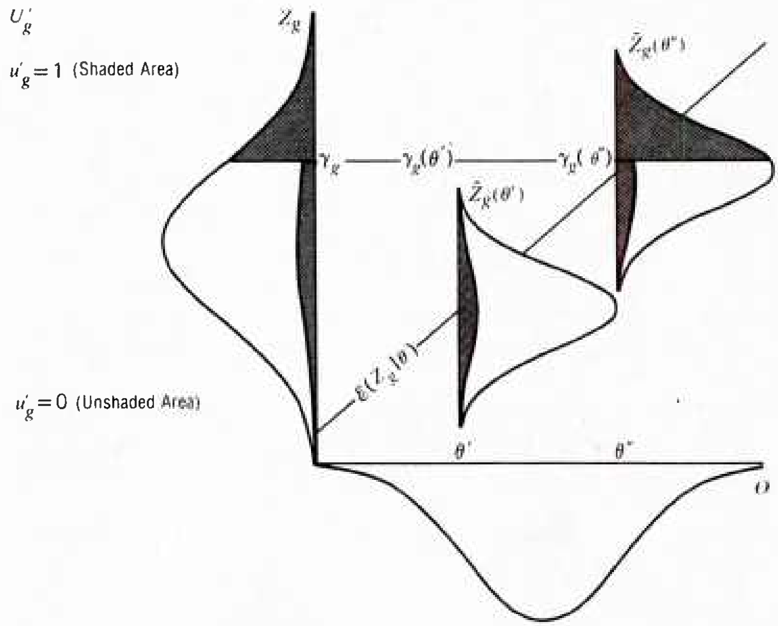

Figure 3.1.1. Hypothetical relations among the item continuum Z_g, the multiple-choice, binary item U'_g, and the latent trait Θ.

Substitutions from (2.1.18) into (3.1.11) now allow one to write

$$P'_g(\theta) = c_g + (1 - c_g)\,\Phi[-\gamma_g(\theta)] = c_g + (1 - c_g)\,\Phi[a_g(\theta - b_g)] \tag{3.1.12}$$

where $\Phi[*]$ is again by definition the cumulative normal distribution function as expressed in (2.1.18).

Note that the probability of answering a multiple-choice item g incorrectly given a fixed value of ability, $Q'_g(\theta)$, is provided by

$$Q'_g(\theta) = (1 - c_g)\Pr[\tilde{Z}_g(\theta) < \gamma_g(\theta)] = (1 - c_g)\int_{-\infty}^{\gamma_g(\theta)} (2\pi)^{-.5}\exp\{-.5[\tilde{\zeta}_g(\theta)]^2\}\,d\tilde{\zeta}_g(\theta) \tag{3.1.13}$$

which after substitutions from (2.1.20) yields

$$Q'_g(\theta) = (1 - c_g)\,Q_g(\theta) = (1 - c_g)\,\Phi[\gamma_g(\theta)] = (1 - c_g)\,\Phi[-a_g(\theta - b_g)], \tag{3.1.14}$$

The conditional probability, $Q'_g(\theta)$, provided in (3.1.13) and (3.1.14) is the complement of the conditional probability, $P'_g(\theta)$, provided in (3.1.10) through (3.1.12), because

$$P'_g(\theta) + Q'_g(\theta) = 1 \tag{3.1.15}$$

which may be verified through the substitution of the second equality in (3.1.14) and the rightmost equality in (3.1.10) into (3.1.15). After these substitutions are made, one may write

$$P_g(\theta) + Q_g(\theta) = 1 \tag{3.1.16}$$

a relationship established in connection with (2.1.20).

The Parameters

Equation (3.1.10) represents an expression for the item characteristic curve, the item true score function T'_g, and the regression of a multiple-choice, binary item U'_g onto Θ. This functional form provides the probability of a correct answer to item g as a function of Θ. The parameters of this expression are a_g, b_g, and c_g, where a_g and b_g are defined in (2.1.12) and (2.1.13) and c_g is the coefficient of guessing, that is, the rate at which successful guessing occurs among individuals who do not recognize the correct alternative.

Figure 2.1.1 for a free-response item g may be compared with Figure 3.1.1 for a multiple-choice item g. Notice that the dichotomization of the $\bar{Z}_g(\theta')$ and $\bar{Z}_g(\theta'')$ at $\gamma_g(\theta')$ and $\gamma_g(\theta'')$, respectively, occurs in both figures. Actually, the shaded areas below $\gamma_g(\theta')$ and $\gamma_g(\theta'')$ in Figure 3.1.1 for the multiple-choice case represent the only departure from a dichotomization of the conditional distributions in determining correct and incorrect answers. Thus, there is a similarity between the two- and three-parameter submodels. This similarity is illustrated in Figure 3.1.2, which presents the item characteristic curve (solid function) for a multiple-choice item g. Also, in the figure, a dashed function is presented. This dashed function is actually a two-parameter curve representing the probability that the correct alternative will be recognized given θ. The solid regression function, $P'_g(\theta)$, can be viewed as the curve that connects the conditional proportions that are the output of (3.1.10) at each level of Θ. The dashed function is obtained from the leftmost term in the rightmost equality in (3.1.10), namely, $P_g(\theta)$, where $P_g(\theta)$ is based, as in the free-response case, on the parameters a_g and b_g. Thus, the earlier discussion of these parameters is still fully pertinent. What is new is the contribution of guessing, which can be obtained from the rightmost term on the right hand side of (3.1.10), namely, the product $c_g Q_g(\theta)$. As pictured in Figure 3.1.2, this contribution is represented by the vertical distance between the solid and dashed functions for a given value θ. Notice that the coefficient of guessing, c_g, is constant with respect to the random variable Θ, but that the contribution of guessing diminishes as θ increases. Guessing is only invoked when the correct alternative is not recognized. Since the probability of recognizing the correct alternative is an increasing function of latent ability, the product of its complement and a constant is a diminishing function of latent ability. Notice in Figure 3.1.2 that c_g is the lower asymptote of the item characteristic curve as measured on the ordinate. More formally, then, c_g is defined as

$$c_g = \lim_{\theta \to -\infty} P'_g(\theta), \tag{3.1.17}$$

the limiting value (lower asymptote) of $P'_g(\theta)$ as θ approaches negative infinity.

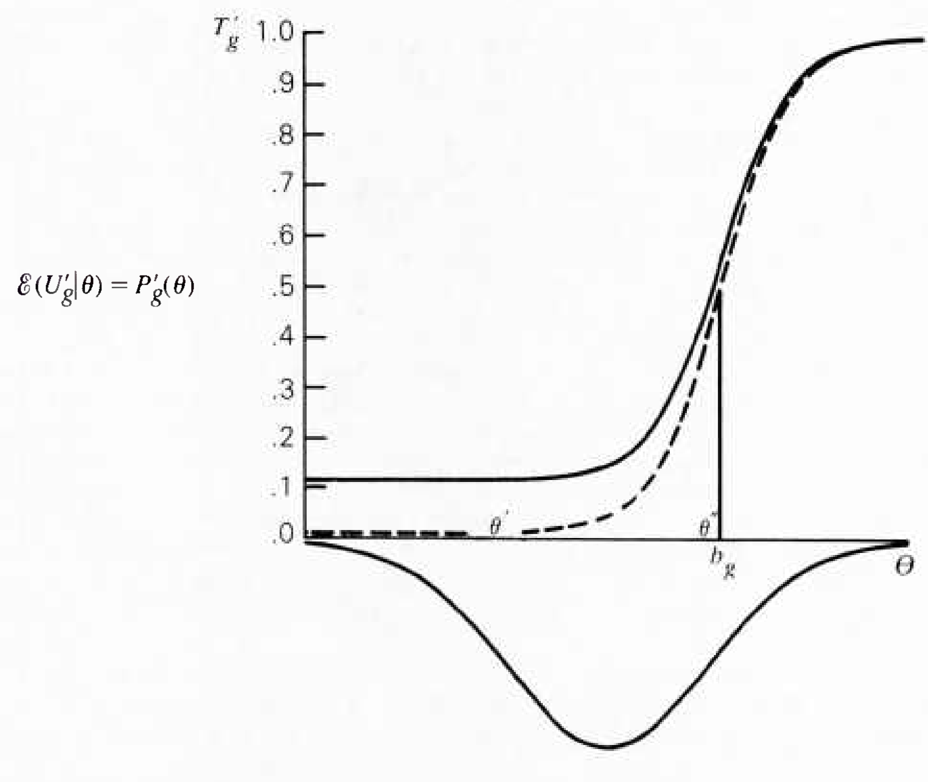

Figure 3.1.2. The item characteristic curve or the regression (solid function) of a binary, multiple-choice item U'_g on the latent trait Θ.

At this juncture, some numerical examples are helpful. These examples will highlight the relationships between Figure 3.1.1 and Figure 3.1.2 for item g given its parameters.

Example 3.1.1. The slope of the regression of Z_g onto Θ in Figure 3.1.1 is .86. This value is also $\rho(Z_g, \Theta)$ because both Z_g and Θ are depicted as distributed with a mean of zero and a variance of unity. Thus item discriminatory power, a_g, as given by (2.1.12) is 1.6853. In Figure 3.1.1, the ordinate and abscissa are both portrayed for the range of values from -3.0 to $+3.0$. The projection of the point of intersection of the line segment extending horizontally through γ_g and the line of regression, $\mathscr{E}(\Theta \mid \zeta_g)$, occurs at a value of 1.1628. Thus, b_g is equal to 1.1628. The distributions of Z_g given θ are symmetric about their means which coincide with the line of regression, $\mathscr{E}(Z_g \mid \theta)$. Hence, at this intersection where θ equals b_g, $P_g(b_g)$, the probability of recognizing the correct alternative to item g, is .5. This probability may be ascertained by examining the dashed function in Figure 3.1.2. The probability of a correct answer to this multiple-choice item g at θ equal to b_g, $P_g'(b_g)$, is .555 as may be judged through the examination of the solid function in Figure 3.1.2. The lower asymptote of the item characteristic curve for this item g, as may be verified by inspection of Figure 3.1.2, is .11. Since c_g is .11 and $P_g(\theta)$ is .50 at θ equal to b_g, $P_g'(b_g)$ as evaluated by (3.1.11) is .555.

Example 3.1.2. In Figures 3.1.1 and 3.1.2, θ' has the value of -1.10. When $\gamma_g(\theta)$ of (2.1.15) is evaluated at θ', $\gamma_g(\theta')$ is found to equal 3.8135. Note that the standardized conditional distribution of $\tilde{Z}_g(\theta')$, as well as that of $\tilde{Z}_g(\theta'')$, is displayed in Figure 3.1.2 as ranging in value from -3.0 to $+3.0$. This range of values accounts for .9973 of the probability or area under the curve. Thus, the area above $\gamma(\theta')$ on $\tilde{Z}_g(\theta')$, the standardized conditional distribution of Z_g given θ', is provided by (2.1.18) as .00. This value of $P_g(\theta')$ is indicated by the relationship between $\gamma_g(\theta')$, the point of dichotomization on $\tilde{Z}_g(\theta')$, and $\tilde{Z}_g(\theta')$. The probability of recognizing the correct alternative to item g given θ' is negligible. However, the probability of obtaining a correct answer to item g given θ', $P_g'(\theta')$, is not negligible. Since c_g equals .11 and $P_g(\theta')$ equals .00, $P_g'(\theta)$, when evaluated by (3.1.11) at θ', is found to be .11. This probability is represented in Figure 3.1.2 by the shaded area in the standardized conditional distribution of $\tilde{Z}_g(\theta')$. This shaded area occurs in the effective range for $\tilde{Z}_g(\theta')$ which is below the point of dichotomization on $\tilde{Z}_g(\theta')$, $\gamma_g(\theta')$.

The value of .11 for $P_g'(\theta')$ represents the probability of obtaining a correct answer to item g given a standard ability score of -1.10. Thus, a point may be placed in Figure 3.1.2 with the coordinates of .11 on the ordinate, $P_g'(\theta)$, and θ', -1.10, on the abscissa Θ.

Example 3.1.3. In Figures 3.1.1 and 3.1.2, θ'' has the value of 1.00. When $\gamma_g(\theta)$ of (2.1.15) is evaluated at θ'', it is found that $\gamma_g(\theta'')$ equals .2744. The area above $\gamma_g(\theta'')$ on $\tilde{Z}_g(\theta'')$, the standardized conditional distribution of Z_g given θ'', is provided by (2.1.18) as .3919. This value of $P_g(\theta'')$, the probability of recognizing the correct alternative to item g, is portrayed in Figure 3.1.1 as the shaded area in the standardized conditional distribution of $\tilde{Z}_g(\theta'')$ above the point of dichotomization $\gamma_g(\theta'')$. The probability of obtaining a correct answer to item g given θ'' is obtained from (3.1.11). Given c_g equal to .11 and $\gamma_g(\theta'')$ equal to .3919, an evaluation of (3.1.11) yields .4588 for this probability, $P_g'(\theta'')$. This probability is represented in Figure 3.1.1 by the shaded area in the standardized conditional distribution of $\tilde{Z}_g(\theta'')$ which occurs both above and below the point of dichotomization, $\gamma_g(\theta'')$. In this context, $\gamma_g(\theta)$ might be referred to as the threshold of recognition given θ. As discussed earlier, obtaining the correct answer to multiple-choice item g, however, is not the same as recognizing the correct alternative. The selection of the correct alternative or obtaining a correct answer to multiple-choice item g occurs through successful guessing when the threshold of recognition on $\tilde{Z}_g(\theta)$ has not been reached.

The value of .4588 for $P_g'(\theta'')$ represents the probability of obtaining a correct answer to item g given a standard ability score of 1.00. Thus, a point may be placed in Figure 3.1.2 with the coordinates of .4588 on the ordinate, $P_g'(\theta'')$, and θ'', 1.00, on the abscissa Θ.

The item characteristic curve of Figure 3.1.2 can be constructed by repeating the numerical process illustrated in Examples 3.1.2 and 3.1.3 for a sufficient number of points equally spaced on the abscissa, Θ. It is useful to be able to examine the item characteristic curves for various items given their parameters. The characteristic curve for a multiple-choice item g, given its parameters, can be conveniently produced for this purpose through the aid of a plotter and the formulation as illustrated in Examples 3.1.2 and 3.1.3 for the arbitrary points θ' and θ''.

3.2. Further Consequences of the Submodel

Given the basic model from which the three-parameter normal ogive submodel is derived, it is known that the joint distribution of Z_g and Θ is still bivariate normal. This condition of bivariate normality obtains even though the phenomenon of success due to guessing on multiple-choice items has been accommodated in this submodel. As in the case of the two-parameter submodel, this condition facilitates the derivations of convenient mathematical expressions. Again, the derivations for these expressions follow from defining relationships within the bivariate normal distribution. In this section, these expressions will be presented and numerically illustrated using multiple-choice item g as portrayed in Figure 3.1.1 for illustrative purposes. Given the three-parameter normal ogive submodel, convenient expressions will be presented for:

1. The unconditional probabilities for the realizations of U'_g.
   (a) The probability of a correct answer to item g, $\Pr(u'_g = 1)$.
   (b) The probability of an incorrect answer to item g, $\Pr(u'_g = 0)$.
2. The conditional means of Z_g given the realizations of U'_g.
   (a) The mean of Z_g given a correct answer to item g, $\mu(Z_g \mid u'_g = 1)$.
   (b) The mean of Z_g given an incorrect answer to item g, $\mu(Z_g \mid u'_g = 0)$.
3. The conditional variances of Z_g given the realizations of U_g.
   (a) The variance of Z_g given a correct answer to item g, $\sigma^2(Z_g \mid u'_g = 1)$.
   (b) The variance of Z_g given an incorrect answer to item g, $\sigma^2(Z_g \mid u'_g = 0)$.
4. The least squares estimators of ability given the realizations of U'_g.
   (a) Specific case.
      (1) The mean of Θ given a correct answer to item g, $\mu(\Theta \mid u'_g = 1)$.
      (2) The mean of Θ given an incorrect answer to item g, $\mu(\Theta \mid u'_g = 0)$.
   (b) General Case.
      (1) The mean of Θ^* given a correct answer to item g, $\mu(\Theta^* \mid u'_g = 1)$.
      (2) The mean of Θ^* given an incorrect answer to item g, $\mu(\Theta^* \mid u'_g = 0)$.

These expressions will be derived in Section 3.3. For those readers seeking a general understanding, Section 3.3 may be omitted without loss of continuity.

The Unconditional Probabilities for the Realizations of U'_g

The binary random variable U'_g can realize only one of two possible values. In the instance of a correct answer to a multiple-choice item g, u'_g, the realization of U'_g, equals *one*. In the instance of an incorrect answer to a multiple-choice item g, the realization of U'_g, u'_g equals *zero*.

The Probability of a Correct Answer to Item g, $\Pr(u'_g = 1)$. This unconditional probability is designated P'_g. Most commonly, it is known as the p-value for multiple-choice item g. A convenient expression for this probability is

$$\Pr(u'_g = 1) = P'_g = c_g + (1 - c_g)\,P_g = c_g + (1 - c_g)\,\Phi[-\gamma_g], \tag{3.2.1}$$

where c_g is the guessing coefficient as defined in (3.1.17) and P_g, which equals $\Phi[-\gamma_g]$, is the probability of recognizing the correct alternative to multiple-choice item g as given by (2.2.1). Notice that γ_g, as required in an evaluation of (2.2.1), is identically defined under both the two- and three-parameter normal ogive submodels. This situation exists because γ_g as seen in (2.2.2) is determined by $\rho Z_g,\Theta)$ and b_g, both of which are identically defined under these submodels. The difference under these submodels resides in the interpretation of the probability P_g. In the previous context, this unconditional probability was that of obtaining a correct answer to a free-response item g. In the present context, this unconditional probability represents that of recognizing the correct alternative to a multiple-choice item g.

Example 3.2.1. It is known from Example 3.1.1 that $\rho(Z_g,\Theta)$, the correlation between the continuous variables Z_g and Θ, is .86, and that b_g, the item difficulty parameter, is 1.1628. Thus γ_g, the point of dichotomization on the continuous variable Z_g, is found to be 1.00 when (2.2.2) is evaluated. Hence, P_g, the unconditional probability of recognizing the correct answer to multiple-choice item g as given by (2.2.1), is .1587. Since P_g is .1587 and c_g, as determined in Example 3.1.1, is .11, P'_g as evaluated by (3.2.1) is .2512. The unconditional probability P'_g is that of obtaining a correct answer to multiple-choice item g.

The value of P_g, .1587, represents the size of the shaded area above γ_g in the marginal distribution of Z_g as portrayed along the ordinate in Figure 3.1.1. If γ_g is considered the threshold of recognition on Z_g, then the size of this shaded area represents the probability of recognizing the correct alternative to multiple-choice item g. The value of P'_g, .2512, represents the size of the combined shaded areas both above and below γ_g in the marginal distribution of Z_g as depicted along the ordinate in Figure 3.1.1. The size of these shaded areas represents the probability of obtaining a correct answer to multiple-choice item g.

Since obtaining a correct answer to multiple-choice item g is not fully determined by a dichotomization of a normally distributed random variable, it is clearly a contradiction given this submodel to estimate $\rho(Z_g,\Theta)$, the correlation between the continuous variables Z_g and Θ, through the use of the biserial correlation. Furthermore, since obtaining correct answers to any two multiple-choice items g and h is not fully determined by the dichotomizations of the continuous variables Z_g and Z_h, it is clearly a contradiction given this submodel to estimate $\rho(Z_g,Z_h)$, the correlation between the continuous variables Z_g and Z_h, through the use of the tetrachoric correlation. The use of the tetrachoric correlation with multiple-choice items in factor-analytic studies is, unfortunately, widespread. This practice is inappropriate because the submodel considered here has been found to be valid for multiple-choice items (Urry, 1977).

The Probability of an Incorrect Answer to Item g, Pr $(u'_g = 0)$. This unconditional probability is designated as Q'_g. A convenient expression for this probability is

$$\Pr(u'_g = 0) = Q'_g = (1 - c_g)\, Q_g = (1 - c_g)\, \Phi[\gamma_g], \tag{3.2.2}$$

where c_g is the coefficient of guessing as defined in (3.1.17) and Q_g, which equals $\Phi[\gamma_g]$, the probability of not recognizing the correct alternative to multiple-choice item g, is given by (2.2.3). In the context of the two-parameter normal ogive submodel, this unconditional probability was that of obtaining an incorrect answer to a free-response item g. In the present context, this unconditional probability is that of not recognizing the correct alternative to a multiple-choice item g. Since the rate of successful guessing on item g is c_g, $(1 - c_g)$ is the rate of unsuccessful guessing. Thus, the product of $(1 - c_g)$ and Q_g as given in (3.2.2), represents the probability of not recognizing the correct alternative to multiple-choice item and guessing incorrectly. That is to say, this product, Q'_g, represents the unconditional probability of an incorrect answer to a multiple-choice item g.

Example 3.2.2. It is known from Example 3.2.1 that γ_g, the point of dichotomization on Z_g for item g of Figure 3.1.1, is 1.00. Hence Q_g, the unconditional probability of not recognizing the correct alternative to multiple-choice item g as evaluated with (2.2.3), is .8413. Since Q_g is .8413 and c_g, as determined in Example 3.1.1, is .11, Q'_g as evaluated through the use of (3.2.2) is .7488. Notice that this value is the complement of P'_g or .2512 as is Q_g or .8413 the complement of P_g or .1587.

The value of Q_g represents the size of the area both shaded and unshaded below γ_g in the marginal distribution of Z_g as portrayed along the ordinate in Figure 3.1.1. The size of this area represents the probability of being below the threshold of recognition on the continuous variable Z_g. The value of Q'_g or .7488, represents the size of the unshaded area below γ_g in the marginal distribution of Z_g as depicted along the ordinate in Figure 3.1.1. This unconditional probability is that of obtaining an incorrect answer to multiple-choice item g, that is to say, the probability of not recognizing the correct alternative to multiple-choice item g and guessing unsuccessfully.

If γ_g were actually a cut score on an observed continuous variable Z_g, and that subpopulation normally rejected were also selected at random at a rate c_g, then P'_g would be the selection ratio or the probability of being selected, and Q'_g would be the probability of being rejected. In this submodel, there also exists an analogous relationship between the binary scores on a multiple-choice item g, either a *one* or a *zero*, and this particular method of selection or rejection on an observed continuous variable. This analogy, as well as the one drawn earlier with respect to the two-parameter normal ogive submodel, provides the basis for later developments in Chapter 5. In that chapter the tailoring algorithms will be derived through the use of selection theory. This theory is applicable when the method of selection or rejection can be specified.

The Conditional Means of Z_g Given the Realizations of U'_g

The mean of Z_g can assume one of two possible values. When multiple-choice item g is answered correctly, u'_g, the realization of the binary random variable U'_g, equals *one*. Thus the mean of Z_g given a correct answer to multiple-choice item g is designated $\mu(Z_g \mid u'_g = 1)$. When multiple-choice item g is answered incorrectly, u'_g, the realization of the binary random variable U'_g, equals *zero*. Thus the mean of Z_g given an incorrect answer to multiple-choice item g is designated $\mu(Z_g \mid u'_g = 0)$.

The Mean of Z_g Given a Correct Answer to Item g, $\mu(Z_g \mid u'_g = 1)$. A convenient expression for this mean is given by

$$\mu(Z_g \mid u'_g = 1) = (1 - c_g)\,\frac{\phi(\gamma_g)}{P'_g}, \tag{3.2.3}$$

where c_g is the coefficient of guessing as defined in (3.1.17); $\phi(\gamma_g)$ is the density in the standard normal distribution evaluated at γ_g, the point of dichotomization on the continuous variable Z_g, as provided by (2.2.5); and P'_g is the unconditional probability of obtaining a correct answer to multiple-choice item g as given by (3.2.1).

> *Example 3.2.3.* In Example 3.2.1 it was found that γ_g and P'_g for item g as depicted in Figure 3.1.1 were 1.00 and .2512, respectively. Thus $\phi(\gamma_g)$ as evaluated in (2.2.5) is .2420. Since c_g, as determined in Example 3.1.1, is .11, $\phi(\gamma_g)$ is .2420, and P'_g is .2512, an evaluation of $\mu(Z_g \mid u'_g = 1)$ through use of (3.2.3) with these values yields .8574 as the mean of Z_g given a correct answer to multiple-choice item g.
>
> This value represents the mean of the combined shaded areas in the marginal distribution of Z_g as portrayed in Figure 3.1.1 along the ordinate. Remember this distribution is displayed for the interval on Z_g ranging from -3.0 to $+3.0$.

The Mean of Z_g Given an Incorrect Answer to Item g, $\mu(Z_g \mid u'_g = 0)$. A convenient expression for this mean is provided by

$$\mu(Z_g \mid u'_g = 0) = -\,\frac{\phi(\gamma_g)}{Q_g}, \tag{3.2.4}$$

where $\phi(\gamma_g)$ is the density in the standard normal distribution evaluated at γ_g, the point of dichotomization on the continuous variable Z_g, as provided by (2.2.5); and Q_g is the unconditional probability of not recognizing the correct alternative to multiple-choice item g as given by (2.2.3). Notice that this mean is obtained through a formula which is algebraically equivalent to that of (2.2.6) for its counterpart under the two-parameter normal ogive submodel. This algebraic equivalence obtains because success due to guessing in the three-parameter case is assumed to occur at random at a rate of c_g on the interval of Z_g from negative infinity to γ_g.

> *Example 3.2.4.* For the item portrayed in Figure 3.1.1, it was found in Example 3.2.3 that $\phi(\gamma_g)$ equalled .2420, and in Example 3.2.2 that Q_g equalled .8413. When (3.2.4) is evaluated for these values of $\phi(\gamma_g)$ and Q_g, it is known that the mean of Z_g for the subpopulation obtaining an incorrect answer to multiple-choice item g, $\mu(Z_g \mid u'_g = 0)$, is $-.2877$. This value, $-.2877$, is the mean of Z_g for the subpopulation represented by the unshaded portion in the marginal distribution of Z_g as portrayed along the ordinate in Figure 3.1.1.

If γ_g were actually a cut score on an observed continuous variable Z_g and complete selection occurred when ζ_g was equal to or greater than γ_g and random selection occurred at a rate c_g when ζ_g was less than γ_g, then $\mu(Z_g \mid u'_g = 1)$ would be the mean of Z_g for the subpopulation explicitly selected on the continuous variable Z_g. Accordingly, $\mu(Z_g \mid u'_g = 0)$ would then be the mean of Z_g for the subpopulation not selected, that is, the subpopulation explicitly rejected on the continuous variable Z_g.

The Conditional Variances of Z_g Given the Realizations of U'_g

These variances of Z_g can assume two possible values. When multiple-choice item g is answered correctly, the realization of the binary random variable U'_g, u'_g, is *one*. Hence the variance of Z_g given a correct answer to a multiple-choice item g is designated as $\sigma^2(Z_g \mid u'_g = 1)$. When multiple-choice item g is answered incorrectly,

u_g', the realization of the binary random variable U_g', is *zero*. Hence the variance of Z_g given an incorrect answer to multiple-choice item g is designated as $\sigma^2(Z_g \mid u_g' = 0)$.

The Variance of Z_g Given a Correct Answer to Item g, $\sigma^2(Z_g \mid u_g' = 1)$. A convenient expression for this variance is provided by

$$\sigma^2(Z_g \mid u_g' = 1) = 1 - \frac{(1 - c_g)\,\phi(\gamma_g)}{P_g'}\left[\frac{(1 - c_g)\,\phi(\gamma_g)}{P_g'} - \gamma_g\right], \tag{3.2.5}$$

where c_g, the coefficient of guessing for item g, is defined by (3.1.17); γ_g, the point of dichotomization on the continuous variable Z_g, is provided by (2.2.2); $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at γ_g, is given by (2.2.5); and P_g', the unconditional probability of a correct answer to multiple-choice item g, is provided by (3.2.1).

Example 3.2.5. From Example 3.2.1 it is known that c_g is .11, γ_g is 1.00, and P_g' is .2512 for item g as depicted in Figure 3.1.1. It is also known for this item from Example 3.2.3 that $\phi(\gamma_g)$ is .2420. Given these values for c_g, γ_g, P_g', and $\phi(\gamma_g)$, an evaluation of (3.2.5) yields 1.1223 as $\sigma^2(Z_g \mid u_g' = 1)$, the variance of Z_g given a correct answer to this item g. Thus, $\sigma(Z_g \mid u_g' = 1)$, the standard deviation of Z_g given a correct answer to this item g, is 1.0594. The value 1.0594 is the standard deviation for the subpopulation represented by the shaded area in the marginal distribution of Z_g as portrayed along the ordinate in Figure 3.1.1.

The Variance of Z_g Given an Incorrect Answer to Item g, $\sigma^2(Z_g \mid u_g' = 0)$. A convenient expression for this variance is given by

$$\sigma^2(Z_g \mid u_g' = 0) = 1 - \frac{\phi(\gamma_g)}{Q_g}\left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g\right], \tag{3.2.6}$$

where γ_g, the point of dichotomization on the continuous variable Z_g, is given by (2.2.2); $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at γ_g, is defined by (2.2.5); and Q_g, the probability of not recognizing the correct alternative to multiple-choice item g, is provided by (2.2.3).

Example 3.2.6. It is known from Example 3.2.1 that γ_g is 1.00 for item g as depicted in Figure 3.1.1. Also, from Example 3.2.2, it is known that Q_g is .8413; and from Example 3.2.3 it is known that $\phi(\gamma_g)$ equals .2420 for this item g. When (3.2.6) is evaluated with these values for γ_g, Q_g, and $\phi(\gamma_g)$, $\sigma^2(Z_g \mid u_g' = 0)$, the variance of Z_g given an incorrect answer to this item g, is .6296. Thus, $\sigma(Z_g \mid u_g' = 0)$, the standard deviation of Z_g given an incorrect answer to this item g, is .7935. The value .7935 is the standard deviation for the subpopulation represented by the unshaded portion of the marginal distribution of Z_g as portrayed along the ordinate in Figure 3.1.1.

Notice that this standard deviation is obtained through a formula that is equivalent to the square root of (2.2.8), its counterpart under the two-parameter normal ogive submodel. This algebraic equivalence holds because the subpopulation obtaining an incorrect answer to a multiple-choice item g is assumed to result from a random depletion of the subpopulation on the interval from negative infinity to γ_g on the continuous variable Z_g. In the submodel for free-response items no depletion is assumed for the subpopulation on this interval. Since the depletion is random when it occurs, the standard deviation given an incorrect answer to item g is invariant with respect to these submodels where guessing is either accommodated or not accommodated.

Least Squares Estimators of Ability Given the Realizations of U_g'

The specific and general cases are considered. In the specific case, ability, Θ, has a mean, $\mu(\Theta)$, of zero and a variance, $\sigma^2(\Theta)$, of unity. In the general case, ability, Θ^*, has a mean, $\mu(\Theta^*)$, and a variance, $\sigma^2(\Theta^*)$, that are prescribed by the practitioner for convenience in applications.

Specific case. In this case, the estimators of ability, Θ, depend on the realizations of the binary random variable U_g' or the correctness or incorrectness of the answer to multiple-choice item g. When the answer is correct, the estimator of ability is the mean of Θ for the subpopulation that would obtain a correct answer on item g. When the answer is incorrect, the estimator of ability, Θ, is the mean of Θ for the subpopulation that would obtain an incorrect answer on item g.

Specific Case: The Mean of Θ Given a Correct Answer to Item g, $\mu(\Theta \mid u'_g = 1)$. This estimator is a least squares estimator because this mean provides that value of Θ about which the sum of squared discrepancies is minimized for the variable of ability given a correct answer to a multiple-choice item g. A convenient expression for this estimator of ability is provided by

$$\mu(\Theta \mid u'_g = 1) = \rho(Z_g, \Theta)\,(1 - c_g)\,\frac{\phi(\gamma_g)}{P'_g}, \tag{3.2.7}$$

where $\rho(Z_g,\Theta)$, the correlation between the continuous variables Z_g and Θ, is known when a_g, the item parameter of discriminatory power, is known through the use of (2.2.10); c_g, the coefficient of guessing for item g, is defined by (3.1.17); γ_g, the point of dichotomization on the continuous variable Z_g, is known when $\rho(Z_g,\Theta)$ and b_g, the item parameter of difficulty, are known through the use of (2.2.2); $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at γ_g, is defined by (2.2.5); and P'_g, the unconditional probability of a correct answer to a multiple-choice item g, is given by (3.2.1).

Example 3.2.7. In Example 3.1.1, it was found for item g as depicted in Figure 3.1.1 that the parameter of discriminatory power for this item is 1.6853. When (2.2.10) is evaluated for this value of a_g, $\rho(Z_g,\Theta)$ is found to be .86, which was the value obtained earlier through the inspection of Figure 3.1.1 in Example 3.1.1. In this example, c_g was determined to be .11. In Example 3.2.1, γ_g and P'_g were found to be 1.00 and .2512, respectively; and in Example 3.2.3, $\phi(\gamma_g)$ was determined as .2420. With these values for $\rho(Z_g,\Theta)$, c_g, γ_g, P'_g, and $\phi(\gamma_g)$, an evaluation of (3.2.7) yields .7374 as the value of the mean of Θ given a correct answer to this multiple-choice item g. The value .7374 represents the mean of the distribution depicted in Figure 3.2.1.

The distribution in this figure is readily constructed. The value of the ordinate in this distribution, $f(\theta \mid u'_g = 1)$, at a particular value of the abscissa, Θ, can be obtained from

$$f(\theta \mid u'_g = 1) = P'_g(\theta)\,\phi(\theta) \tag{3.2.8}$$

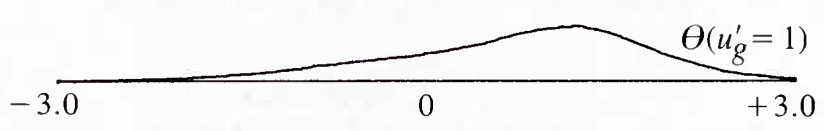

Figure 3.2.1. The distribution of ability given u'_g equal to *one*, $\Theta(u'_g = 1)$, or, identically, the distribution of Θ resulting from incidental selection on Θ due to explicit selection on the continuous variable Z_g, where γ_g is the cut score at or above which selection is complete and below which selection occurs at random at a rate c_g.

for the range of values of Θ from negative to positive infinity. At any particular value θ, the ordinate is the product of $P_g'(\theta)$, the probability of a correct answer to a multiple-choice item g as provided by (3.1.12), and $\phi(\theta)$, the density function for the assumed distribution of ability as defined by (2.2.12). When the values of the ordinate as obtained from (3.2.8) have been plotted with respect to the abscissa for the range of values from -3.0 to $+3.0$, the curve delineating the distribution in Figure 3.2.1 is the result.

From integral calculus it is known that the area under the curve described by (3.2.8) is P_g', the unconditional probability of a correct answer to a multiple-choice item g. Thus, the density function for the distribution of $\Theta(u_g' = 1)$, $\phi(\theta \mid u_g' = 1)$, is provided by

$$\phi(\theta \mid u_g' = 1) = \frac{f(\theta \mid u_g' = 1)}{P_g'}, \tag{3.2.9}$$

where the area under the curve described by this equation is unity. In Equation (3.2.9), $f(\theta \mid u_g' = 1)$ is the ordinate of the distribution of $\Theta(u_g' = 1)$ as given by (3.2.8) and P_g' is the unconditional probability of a correct answer to a multiple-choice item g as provided by (3.2.1).

Earlier it was discussed that (2.1.11) and (2.1.15) in effect translated the point of dichotomization on the continuous variable Z_g into the $\gamma_g(\theta)$, the points of dichotomization on the standardized conditional distributions of Z_g given θ, the $\bar{Z}_g(\theta)$. This translation allowed the use of the cumulative normal distribution function, $\Phi[*]$, in (3.1.12) to obtain $P_g'(\theta)$, the probability of a correct answer to a multiple-choice item g given θ. Now if γ_g were actually a cut score on an observed continuous variable Z_g, then the $\gamma_g(\theta)$ would be the corresponding cut scores on the standardized conditional distributions of Z_g given θ, the $\bar{Z}(\theta)$. Under this interpretation, $P_g'(\theta)$ would be the probability of being selected given θ. In this situation, selection is complete at and above a cut score of $\gamma_g(\theta)$ and occurs at random at a rate c_g below a cut score of $\gamma_g(\theta)$. As a consequence of this interpretation, the distribution depicted in Figure 3.2.1 as generated from (3.2.8) would be the distribution of Θ resulting from incidental selection on the continuous variable Θ due to explicit selection on an observed continuous variable Z_g.

Specific Case: The Mean of Θ Given an Incorrect Answer to Item g, $\mu(\Theta \mid u_g' = 0)$. This estimator is a least squares estimator because this mean provides that value of Θ about which the sum of squared discrepancies is minimized for the variable of ability given an incorrect answer to a multiple-choice item g. A convenient expression for this estimator of ability is given by

$$\mu(\Theta \mid u_g' = 0) = -\rho(Z_g, \Theta)\frac{\phi(\gamma_g)}{Q_g} \tag{3.2.10}$$

where $\rho(Z_g, \Theta)$, the correlation between the continuous variables Z_g and Θ, is known when a_g, the item parameter of discriminatory power, is known through the use of (2.2.10); γ_g, the point of dichotomization of the continuous variable Z_g, is known when $\rho(Z_g, \Theta)$ and b_g, the difficulty parameter for item g, are known through the use of (2.2.2); $\phi(\gamma_g)$, the density in the standard normal distribution evaluated at γ_g, is defined by (2.2.5); and Q_g, the unconditional probability of not recognizing the correct alternative to a multiple-choice item g, is provided by (2.2.3).

> *Example 3.2.8.* In Example 3.1.1 $\rho(Z_g, \Theta)$ was found to be .86 for item g as portrayed in Figure 3.1.1. The density in the standard normal distribution evaluated at γ_g, $\phi(\gamma_g)$, was found to be . 2420; and Q_g, the unconditional probability of not recognizing the correct alternative to this item g, was found to be .8413. When (3.2.10) is evaluated for these values of $\rho(Z_g, \Theta)$, $\phi(\gamma_g)$, and Q_g, $\mu(\Theta \mid u_g = 0)$, the mean of Θ given an incorrect answer to this multiple-choice item g, is known to be $-.2474$. The value $-.2474$ represents the mean of the distribution portrayed in Figure 3.2.2.

The distribution of Θ given an incorrect answer to multiple-choice item g, as presented in this figure, is easily constructed. At a particular value of the abscissa, Θ, the ordinate of this distribution, $f(\theta \mid u_g' = 0)$ can be obtained from

$$f(\theta \mid u_g' = 0) = Q_g'(\theta)\,\phi(\theta) \tag{3.2.11}$$

for the range of values for Θ from negative to positive infinity. At any particular value θ, the ordinate is the product of $Q_g'(\theta)$, the probability of an incorrect answer to a multiple-choice item g as provided by (3.1.14), and $\phi(\theta)$, the density function for the assumed distribution of ability as defined by (2.2.12). When the values

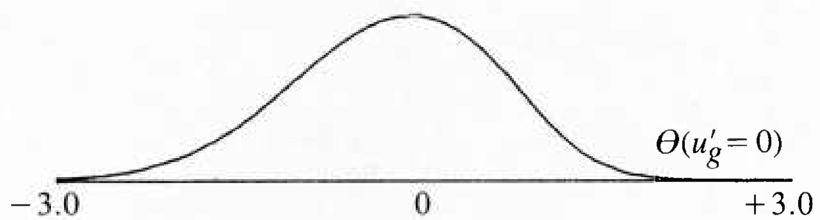

Figure 3.2.2. The distribution of ability given u'_g equal to *zero*, $\Theta(u'_g = 0)$, or, identically, the distribution of Θ resulting from incidental rejection on Θ due to explicit rejection on the continuous variable Z_g, where γ_g is the cut score below which rejection occurs at random at a rate $(1 - c_g)$.

of the ordinate as obtained from (3.2.11) have been plotted with reference to the abscissa for the range of values from -3.0 to $+3.0$, the curve describing the distribution in Figure 3.2.2 is the result.

From integral calculus it is known that the area under the curve delineated by (3.2.11) is Q'_g, the unconditional probability of an incorrect answer to a multiple-choice item g. Thus, the density function for the distribution of $\Theta(u'_g = 0)$, $\phi(\theta \mid u'_g = 0)$, is given by

$$\phi(\theta \mid u'_g = 0) = \frac{f(\theta \mid u'_g = 0)}{Q'_g} \tag{3.2.12}$$

where the area under the curve described by this equation is unity. In Equation (3.2.12), $f(\theta \mid u'_g = 0)$ is the ordinate of the distribution of $\Theta(u'_g = 0)$ as provided by (3.2.11), and Q'_g is the unconditional probability of an incorrect answer to a multiple-choice item g as given by (3.2.2). Equation (3.2.12) can be simplified to

$$\phi(\theta \mid u'_g = 0) = \frac{Q_g(\theta)\ \phi(\theta)}{Q_g} \tag{3.2.13}$$

after substitution from (3.2.11) into (3.2.12) because of (3.1.14) and (3.2.2). In (3.2.13), $Q_g(\theta)$ is the probability of not recognizing the correct alternative to a multiple-choice item g given θ, as provided by (2.1.20); $\phi(\theta)$ is the density function for the assumed distribution of ability, as defined by (2.2.12); and Q_g is the unconditional probability of not recognizing the correct alternative to a multiple-choice item g, as given by (2.2.3).

It has often been mentioned that (2.1.11) and (2.1.15) translate the point of dichotomization on the continuous variable Z_g into the $\gamma_g(\theta)$, the points of dichotomization on the standardized conditional distributions of Z_g given θ, the $\tilde{Z}_g(\theta)$. If γ_g is considered a cut score on an observed continuous variable Z_g, then the $\gamma_g(\theta)$ would be the corresponding cut scores on the $\tilde{Z}_g(\theta)$. Under this interpretation $Q'_g(\theta)$ would be the probability of being rejected when rejection occurs at random at a rate $(1 - c_g)$ for the subpopulation below $\gamma_g(\theta)$ on $\tilde{Z}_g(\theta)$. This rejection rate is random because depletion on the interval from negative infinity to $\gamma_g(\theta)$ on $\tilde{Z}_g(\theta)$ through lucky guessing is assumed to occur at a rate c_g. As a consequence of this interpretation, the distribution portrayed in Figure 3.2.2 as generated from (3.2.11) would be the distribution of Θ resulting from incidental rejection on the continuous variable Θ due to explicit rejection on an observed continuous variable Z_g.

General Case. In the general case of least squares estimators of ability, the practitioner prescribes the mean and variance of Θ^*, the continuous variable of ability. This prescription may be made in order to obtain estimates of ability yielding a particular mean and variance. A particular mean and variance may be convenient in certain applications. How this prescription is obtained and used is illustrated in Chapter 8.

The parameters a_g, item discriminatory power, and b_g, item difficulty, appropriate for the specific case where the continuous variable of ability has a mean, $\mu(\Theta)$, of zero and a variance, $\sigma^2(\Theta)$, of unity were defined by (2.1.12) and (2.1.13). These parameters require a transformation in order to render them appropriate for the continuous variable Θ^* where the mean, $\mu(\Theta^*)$, and the variance, $\sigma^2(\Theta^*)$, have been prescribed by the practitioner. After $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$ have been prescribed and when a_g and b_g are known, the parameters a_g^*, item discriminatory power, and b_g^*, item difficulty, that are appropriate for the continuous variable Θ^* can be obtained from (2.2.17) and (2.2.18). The third item parameter, c_g^*, the coefficient of guessing appropriate for Θ^*, is obtained from

$$c_g^* = c_g, \tag{3.2.14}$$

because the lower asymptote of the item characteristic curve as defined in (3.1.17) is invariant under a linear transformation of the variable of ability.

When a_g^* is known, $\rho(Z_g,\Theta)$, the correlation between the continuous variables Z_g and Θ^*, is given by (2.2.21); and when a_g^* and b_g^* are known, γ_g, the point of dichotomization on the continuous variable Z_g is provided by (2.2.22). Both $\rho(Z_g,\Theta)$ and γ_g are invariant under a linear transformation of the variable of ability.

General Case: The Mean of Θ^* Given a Correct Answer to Item g, $\mu(\Theta^* \mid u_g' = 1)$. This least squares estimator of ability is conveniently provided by

$$\begin{aligned}\mu(\Theta^* \mid u_g' = 1) &= \mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, \mu(Z_g \mid u_g' = 1)\\ &= \mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, (1 - c_g)\, \frac{\phi(\gamma_g)}{P_g'}.\end{aligned} \tag{3.2.15}$$

In (3.2.15), $\mu(\Theta^*)$ and $\sigma(\Theta^*)$ are the prescribed mean and standard deviation of the continuous variable Θ^*; $\rho(Z_g,\Theta)$ is the correlation between the continuous variables Z_g and Θ^* as provided by (2.2.21); $\mu(Z_g \mid u_g' = 1)$ is the mean of Z_g given a correct answer to a multiple-choice item g as given by (3.2.3); c_g is the coefficient of guessing for a multiple-choice item g as defined in (3.1.17); γ_g is the point of dichotomization on the continuous variable Z_g as provided by (2.2.22); $\phi(\gamma_g)$ is the density in the standard normal distribution evaluated at γ_g as obtained from (2.2.5); and P_g' is the unconditional probability of a correct answer to a multiple-choice item g as provided by (3.2.1).

The values for $\rho(Z_g,\Theta)$, γ_g, $\phi(\gamma_g)$, P_g', and $\mu(Z_g \mid u_g' = 1)$ remain invariant under a linear transformation on the scale of the continuous variable of ability.

Example 3.2.9. In the illustration that will be provided in Chapter 8, it is deemed desirable to obtain ability estimates that will have a mean of 100 and a standard deviation of 20. If the precision of these estimates is held constant for all individuals at the termination of testing and comparable to that achieved with a test reliability, as will be defined and described in Chapter 7, of.90, then the mean, $\mu(\Theta^*)$, must be prescribed as 100 and the variance, $\sigma^2(\Theta^*)$, must be prescribed as 444.4444. The standard deviation must then be 21.0819.

In Example 3.1.1, it was found that a_g for item g as portrayed in Figure 3.1.1 is 1.6853. When a_g^* of (2.2.17) is evaluated given the values of 1.6853 for a_g and 21.0819 for $\sigma(\Theta^*)$, a_g^* is found to be .07994. When $\rho(Z_g,\Theta)$ of (2.2.21) is evaluated with these values of a_g^* and $\sigma(\Theta^*)$, $\rho(Z_g,\Theta)$ is found to be .86 (the identical value that was obtained through the use of (2.2.10) in Example 3.2.7). Evaluating b_g^* of (2.2.18) given the values of 1.1628, as obtained from Example 3.1.1, for b_g, 100 and 21.0819 as prescribed for $\mu(\Theta^*)$ and $\sigma(\Theta^*)$, respectively, b_g^* is found to be 124.5140. In obtaining γ_g from (2.2.22) given the values of .07994 for a_g^*, 124.5140 for b_g^*, 100 for $\mu(\Theta^*)$, and 444.4444 for $\sigma^2(\Theta^*)$, γ_g is found to be 1.00 (the identical value that was obtained for γ_g through the use of (2.2.2) in Example 3.2.1). Since γ_g is still 1.00, $\phi(\gamma_g)$ has already been determined. Given this value for γ_g, $\phi(\gamma_g)$ was found to be .2420 in Example 3.2.3. Since γ_g is still 1.00 and c_g as found in Example 3.1.1 is .11, P_g' has also been previously determined. Given these values for γ_g and c_g, P_g' was found to be .2512 in Example 3.2.1. Since

c_g, $\phi(\gamma_g)$, and P'_g are each invariant under a linear transformation on the scale of the continuous variable of ability, $\mu(Z_g \mid u'_g = 1)$, as determined by (3.2.3) in Example 3.2.3, is still .8574.

The least squares estimator of ability, $\mu(\Theta^* \mid u'_g = 1)$, can now be evaluated through the use of (3.2.15). In this situation for this multiple-choice item g, it is known that $\mu(\Theta^*)$ is 100, $\rho(Z_g,\Theta)$ is .86, $\sigma(\Theta^*)$ is 21.0819, and $\mu(Z_g \mid u'_g = 1)$ is .8574. When $\mu(\Theta^* \mid u'_g = 1)$ of (3.2.15) is evaluated given these values, the least squares estimator of ability is found to be 115.5450. The value 115.5450 represents the mean of the distribution depicted in Figure 3.2.1 after a linear transformation of Θ into Θ^*. This transformation can be effected pictorially by merely changing the values along the abscissa from -3.0 to 36.7543, from 0 to 100, and from $+3.0$ to 163.2457. With this transformation on the scale of ability, the distribution of Θ^* given a correct answer or u'_g equal to *one*, $\Theta^*(u'_g = 1)$, is analogous to the distribution of ability after incidental selection on Θ^* resulting from explicit selection on an observed continuous variable Z_g where γ_g is the cut score at and above which selection is complete and below which selection occurs at random at a rate c_g.

General Case: The Mean of Θ^* Given an Incorrect Answer to Item g, $\mu(\Theta^* \mid u'_g = 0)$. This least squares estimator of ability is conveniently provided by

$$\begin{aligned}\mu(\Theta^* \mid u'_g = 0) &= \mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, \mu(Z_g \mid u'_g = 0)\\ &= \mu(\Theta^*) - \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, \frac{\phi(\gamma_g)}{Q_g}. \qquad (3.2.16)\end{aligned}$$

In (3.2.16), $\mu(\Theta^*)$ and $\sigma(\Theta^*)$ are the prescribed mean and standard deviation of the continuous variable Θ^*; $\rho(Z_g,\Theta)$ is the correlation between the continuous variables Z_g and Θ^*, as provided by (2.2.21); $\mu(Z_g \mid u'_g = 0)$ is the mean of Z_g given an incorrect answer to a multiple-choice item g as provided by (3.2.4); γ_g is the point of dichotomization on the continuous variable Z_g as obtained from (2.2.22); $\phi(\gamma_g)$ is the density in the standard normal distribution evaluated at γ_g as defined in (2.2.5); and Q_g is the probability of not recognizing the correct alternative to a multiple-choice item g as provided by (2.2.3). The values for $\rho(Z_g,\Theta)$, γ_g, $\phi(\gamma_g)$, Q_g, and $\mu(Z_g \mid u'_g = 0)$ remain invariant under a linear transformation on the scale of the variable of ability.

Example 3.2.10. As in the instance of Example 3.2.9, it is considered desirable to obtain ability estimates that at the termination of testing will have a mean of 100 and a standard deviation of 20. These respective values can be obtained by terminating the tests for all individuals at a level of precision comparable to that achieved with a test reliability of .90 and prescribing the mean of ability, $\mu(\Theta^*)$, as 100, and the variance of ability, $\sigma^2(\Theta^*)$, as 444.4444, or the standard deviation of ability, $\sigma(\theta^*)$, as 21.0819.

For this situation, $\rho(Z_g,\Theta)$, γ_g, and $\phi(\gamma_g)$ were found to be .86, 1.00, and .2420, respectively, in Example 3.2.9. Since γ_g is invariant under a linear transformation of the continuous variable of ability, so must Q_g, as determined by (2.2.3), be invariant. The unconditional probability of not recognizing the correct alternative to multiple-choice item g, Q_g, was found to be .8413 in Example 3.2.2. The mean of Z_g given an incorrect answer to multiple-choice item g, $\mu(Z_g \mid u'_g = 0)$, or, identically, $\mu(Z_g \mid u_g = 0)$, is also invariant under a linear transformation on the scale of the continuous variable of ability. This invariance necessarily follows because $\phi(\gamma_g)$ and Q_g in (3.2.4) and (2.2.6) are invariant under such a transformation. In Example 3.2.4, $\mu(Z_g \mid u'_g = 0)$, or, identically, $\mu(Z_g \mid u_g = 0)$, was determined to be $-.2877$.

The least squares estimator, $\mu(\Theta^* \mid u'_g = 0)$, can now be evaluated through the use of (3.2.16). In this situation for this multiple-choice item g, it is known that $\mu(\Theta^*)$ is 100, $\rho(Z_g,\Theta)$ is .86, $\sigma(\Theta^*)$ is 21.0819, and $\mu(Z_g \mid u'_g = 0)$ is $-.2877$. When $\mu(\theta^* \mid u'_g = 0)$ of (3.2.16) is evaluated given these values, the least squares estimator of ability given an incorrect answer to multiple-choice item g is found to be 94.7839. The value 94.7839 represents the mean of the distribution portrayed in Figure 3.2.2 after a linear transformation of Θ into Θ^*. Pictorially, this linear transformation can be effected by merely changing the values along the abscissa from -3.0 to 36.7543, from 0 to 100, and from $+3.0$ to 163.2457. With this transformation on the scale of ability, the distribution of Θ^* given an incorrect answer or u'_g equal to *zero* is analogous to the distribution of ability after incidental rejection on Θ^* resulting from explicit rejection on an observed continuous variable Z_g where γ_g is the cut score below which rejection occurs at random at a rate $(1 - c_g)$. This rate is random because the depletion of the subpopulation through lucky guessing at a rate c_g is under this submodel assumed to be random.

The developments illustrated in Sections 2.2 and 3.2 provide the basis for the estimation of an individual's ability given the correctness or incorrectness of the individual's answer to a free-response or multiple-choice item. In tailored testing with one of these types of items, interest will focus on a sequence of such estimations based on items that have been chosen as most appropriate for that individual. The sequence is terminated when the last estimate of ability is sufficiently precise.

In Chapter 5 the illustrated developments of Sections 2.2 and 3.2 will be extended to provide the basic algorithms for the tailored testing of individuals with either free-response or multiple-choice items. But before this extension takes place, a more complete introduction to the process of tailored testing is in order. Such an introduction is provided in Chapter 4.

3.3. Mathematical Proofs

The mathematical formulation that is derived in this section was presented and numerically illustrated in Section 3.2. The mathematical proofs of this formulation as contained in this section may be omitted by the reader who is seeking a general understanding. The omission of this section will not result in a loss of continuity.

In earlier discussion it was noted that the bivariate normal distribution could be viewed as consisting of infinitely many conditional distributions of Z_g given θ, or as consisting of infinitely many conditional distributions of Θ given ζ_g. After standardization, these conditional distributions were defined as the $\tilde{Z}_g(\theta)$ and the $\tilde{\Theta}(\zeta_g)$, respectively. The standardized conditional distributions of Z_g given θ have been discussed in considerable detail with respect to their relationship with the item characteristic curve of the three-parameter normal ogive submodel. An illustration was provided in abbreviated fashion in Figure 3.1.1. The analogous illustration for the conditional distributions of Θ given ζ_g is provided in Figure 3.3.1. In this figure, the following features are important. The projection of the point of dichotomization, γ_g, across the bivariate normal distribution runs parallel to the distributions of $\Theta(\zeta_g)$. Above the point of dichotomization, γ_g, these distributions remain intact as represented by the shaded area indicating a correct answer to item g, $u'_g = 1$. Below the point of dichotomization, γ_g, a proportion c_g that is random with respect to the subpopulations in the $\Theta(\zeta_g)$ also obtains a correct score,

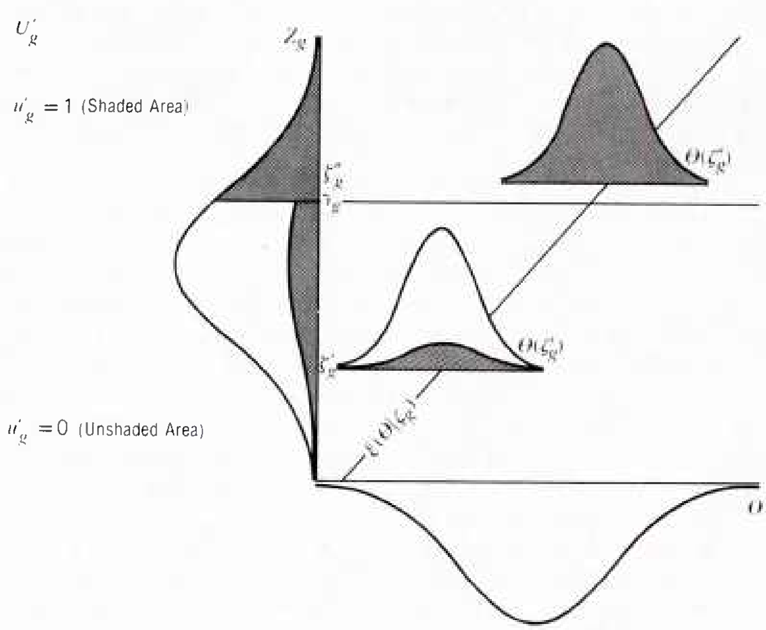

Figure 3.3.1. Hypothetical relations among the item continuum Z_g, the binary multiple-choice item U'_g, the conditional distributions of Θ given ζ_g, and the latent trait Θ.

$u'_g = 1$, in this case through guessing correctly. This condition is also represented by the shaded area in the conditional distribution of $\Theta(\zeta')$ located below γ_g. Also below the point of dichotomization, γ_g, the remaining proportion of this subpopulation, that is $(1 - c_g)$, obtains an incorrect answer to item g, $u'_g = 0$, in this case through guessing incorrectly. This condition is represented by the unshaded area in the conditional distribution of $\Theta(\zeta')$ located below the point of dichotomization, γ_g. Because successful guessing is assumed random on the interval of Z_g from negative infinity to γ_g, it is by extension assumed random with respect to the conditional distributions located below the point of dichotomization, γ_g. Thus the means and variances of the conditional distributions of $\Theta(\zeta_g)$ are invariant under the imposition of the further condition as to whether item g was answered correctly or incorrectly. These relationships become important when selection theory is considered in Chapter 5. These relationships can be sharply contrasted with those presented in Figure 3.1.1, where the binary scoring of item g involves a bisection of the conditional distributions of $Z_g(\theta)$ which, along with considerations of success due to guessing, gives rise to the item characteristic curve as depicted in Figure 3.1.2.

Similarities between the two- and three-parameter normal ogive submodels have often been noted. These similarities will facilitate the developments within this section. Consequently, the present developments will frequently refer to those of Section 2.3 for the two-parameter normal ogive submodel. Several convenient expressions will be derived in the present section. Again, the point of departure for these derivations will be the defining relationships within the bivariate normal distribution. For the initial stage of derivation, the reader may find Figure 2.3.2 helpful in visualizing the particular defining relationship that serves as the basis for the derivation. During this process, the reader should keep in mind the role of guessing as delineated for this submodel.

The Unconditional Probabilities for the Realizations of U'_g

The binary random variable U'_g can realize one of two possible values, either a *one*, or a *zero*, indicating a correct or an incorrect answer, respectively, to a multiple-choice item g. Expressions will now be derived for the probabilities of these two possible outcomes.

The Probability of a Correct Answer to Item g, $\Pr(u'_g = 1)$. This probability is commonly referred to as the p-value for item g. It is designated as merely P'_g. By definition it is then known that

$$\begin{aligned} P_g = \Pr(u'_g = 1) &= \Pr(Z_g \geqslant \gamma_g) + c_g \Pr(Z_g < \gamma_g) \\ &= \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \phi(\zeta_g, \theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \phi(\zeta_g, \theta)\, d\theta\, d\zeta_g, \end{aligned} \tag{3.3.1}$$

where substitutions from (2.3.26) and (2.3.32) provide

$$P'_g = P_g + c_g Q_g. \tag{3.3.2}$$

Upon noting that Q_g is the complement of P_g, one may write

$$P'_g = P_g + c_g(1 - P_g) \tag{3.3.3}$$

which simplifies to

$$P'_g = c_g + (1 - c_g) P_g. \tag{3.3.4}$$

Substitutions from (2.3.31) into (3.3.4) now provide the result

$$\begin{aligned} P'_g &= c_g + (1 - c_g) \int_{\gamma_g}^{\infty} (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g \\ &= c_g + (1 - c_g) \int_{-\infty}^{-\gamma_g} (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g \\ &= c_g + (1 - c_g)\, \Phi[-\gamma_g], \end{aligned} \tag{3.3.5}$$

where $\Phi[*]$ is by definition the cumulative normal distribution function. The equality of integrals in (3.3.5) is due, again, to the symmetry of the normal distribution. The probability of a correct answer to item g, P'_g, is pictorially represented by the shaded area in the marginal distribution of Z_g in Figure 3.1.1.

The Probability of an Incorrect Answer to Item g, $\Pr(u'_g = 0)$. As will become evident, this probability is the complement of P'_g or $(1 - P'_g)$, because the area in the standard normal distribution is unity. This probability is designated as Q'_g.

By definition, it is known that

$$Q'_g = \Pr(u'_g = 0) = (1 - c_g) \Pr(Z_g < \gamma_g)$$
$$= (1 - c_g) \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{3.3.6}$$

where a substitution from (2.3.32) yields

$$Q'_g = (1 - c_g)\, Q_g. \tag{3.3.7}$$

Substitutions from (2.3.35) into (3.3.7) now provide

$$Q'_g = (1 - c_g) \int_{-\infty}^{\gamma_g} (2\pi)^{-.5} \exp(-.5\, \zeta_g^2)\, d\zeta_g = (1 - c_g)\, \Phi[\gamma_g], \tag{3.3.8}$$

where $\Phi[*]$ has previously been defined. The probability of an incorrect answer to item g is portrayed in Figure 3.1.1 as the unshaded area in the marginal distribution of Z_g.

In order to ascertain that the unconditional probabilities P'_g and Q'_g are indeed complementary, it must be proved that

$$P'_g + Q'_g = 1. \tag{3.3.9}$$

Substitutions from (3.3.2) and (3.3.7) into (3.3.9) directly yield

$$P_g + Q_g = 1 \tag{3.3.10}$$

in verification of (3.3.9), because P_g as given in (2.3.31) and Q_g as given in (2.3.35) were found to be complementary probabilities accounting for the total area in the standard normal distribution where this total area is unity. The complementary relationship between P'_g, as represented by the shaded area, and Q'_g as represented by the unshaded area, is also pictorially displayed with respect to the normal distribution of Z_g in Figure 3.1.1.

The Conditional Means of Z_g Given the Realizations of U'_g

The mean of Z_g can realize only one of two possible values, contingent upon whether item g was answered correctly or incorrectly. Expressions will now be derived for these means or expected values for the two possible outcomes.

The Mean of Z_g Given a Correct Answer to Item g, $\mu(Z_g \mid u'_g = 1)$. This mean is given by

$$\mu(Z_g \mid u'_g = 1) = \mathscr{E}(Z_g \mid u'_g = 1)$$
$$= \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g\, \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g\, \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{3.3.11}$$

where $\phi^{*\prime}(\zeta_g,\theta)$, the joint density function for the particular composite of probability integrals, is provided by

$$\phi^{*\prime}(\zeta_g,\theta) = \frac{\phi(\zeta_g,\theta)}{P'_g}. \tag{3.3.12}$$

In order to verify that (3.3.12) is indeed the joint density function for the composite of probability integrals, it is essential to show that the particular composite

$$\int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g \tag{3.3.13}$$

equals unity. After a substitution from (3.3.12) into (3.3.13), one may write

$$\frac{1}{P'_g} \left[\int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g \right], \tag{3.3.14}$$

where substitutions from (2.3.26) and (2.3.32) yield

$$\frac{1}{P'_g} (P_g + c_g\, Q_g). \tag{3.3.15}$$

The substitution of (3.3.2) into (3.3.15) now provides

$$\frac{P'_g}{P'_g} \tag{3.3.16}$$

in verification of (3.3.12) as the joint density function, because (3.3.16) is obviously equal to unity.

After a substitution from (3.3.12) into (3.3.11), one may write

$$\begin{aligned} \mu(Z_g \mid u'_g = 1) &= \mathscr{E}(Z_g \mid u'_g = 1) \\ &= \frac{1}{P'_g} \left[\int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g \right], \end{aligned} \tag{3.3.17}$$

where (2.3.40) and (2.3.53) can be rearranged to yield

$$\mu(Z_g \mid u_g = 1) \, P_g = \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g \tag{3.3.18}$$

and

$$\mu(Z_g \mid u_g = 0) \, Q_g = \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g, \tag{3.3.19}$$

respectively, for substitution into (3.3.17). After these substitutions are made, the composited mean may be expressed as

$$\mu(Z_g \mid u'_g = 1) = \left[\frac{P_g}{P'_g} \right] \mu(Z_g \mid u_g = 1) + \left[\frac{c_g Q_g}{P'_g} \right] \mu(Z_g \mid u_g = 0), \tag{3.3.20}$$

where the bracketed weights sum to unity because of (3.3.2) and provide the relative frequencies for the means resulting from the dichotomization of Z_g. These means, of course, are identically those for Z_g in the two-parameter normal ogive submodel. Substitutions from (2.3.48) and (2.3.59) into (3.3.20) now provide the desired result

$$\mu(Z_g \mid u'_g = 1) = (1 - c_g) \frac{\phi(\gamma_g)}{P'_g} \tag{3.3.21}$$

as a convenient expression for the mean of Z_g given a correct answer to a multiple-choice item g.

The Mean of Z_g Given an Incorrect Answer to Item g, $\mu(Z_g \mid u'_g = 0)$. This mean is given by

$$\begin{aligned} \mu(Z_g \mid u'_g = 0) &= \mathscr{E}(Z_g \mid u'_g = 0) \\ &= (1 - c_g) \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g \, \phi^{\dagger\prime}(\zeta_g,\theta) \, d\theta \, d\zeta_g, \end{aligned} \tag{3.3.22}$$

where $\phi^{\dagger\prime}(\zeta_g,\theta)$, the joint density function for the probability integral, is given by

$$\phi^{\dagger\prime}(\zeta_g,\theta) = \frac{\phi(\zeta_g,\theta)}{Q'_g}. \tag{3.3.23}$$

In order to ascertain that (3.3.23) represents the joint density function for the probability integral, one needs to prove that the particular probability integral

$$(1 - c_g) \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \phi^{\dagger\prime}(\zeta_g,\theta) \, d\theta \, d\zeta_g \tag{3.3.24}$$

equals unity. After a substitution from (3.3.23) into (3.3.24), one may write

$$\frac{(1 - c_g)}{Q'_g} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g, \tag{3.3.25}$$

where a substitution from (2.3.32) allows the writing of

$$\frac{(1 - c_g) \, Q_g}{Q'_g}. \tag{3.3.26}$$

The further substitution from (3.3.7) into (3.3.26) now provides

$$\frac{Q'_g}{Q'_g} \tag{3.3.27}$$

in verification of (3.3.23), because (3.3.27) is obviously equal to unity.

The substitution of (3.3.23) into (3.3.22) allows one to write

$$\begin{aligned}\mu(Z_g \mid u'_g = 0) &= \mathscr{E}(Z_g \mid u'_g = 0) \\ &= \frac{(1 - c_g)}{Q'_g} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g,\end{aligned} \tag{3.3.28}$$

where a substitution from (3.3.19) yields

$$\begin{aligned}\mu(Z_g \mid u'_g = 0) &= \mathscr{E}(Z_g \mid u'_g = 0) \\ &= \frac{(1 - c_g)\, Q_g}{Q'_g} \mu(Z_g \mid u_g = 0).\end{aligned} \tag{3.3.29}$$

A substitution from (3.3.7) into (3.3.29) now produces

$$\mu(Z_g \mid u'_g = 0) = \mu(Z_g \mid u_g = 0), \tag{3.3.30}$$

illustrating that the mean of Z_g given an incorrect answer to item g is identically defined for the two- and three-parameter normal ogive submodels as designed for free-response and multiple-choice items, respectively. A substitution from (2.3.59) into (3.3.30) provides the desired result:

$$\mu(Z_g \mid u'g = 0) = -\frac{\phi(\gamma_g)}{Q_g} \tag{3.3.31}$$

as a convenient expression for the mean of Z_g given an incorrect answer to a multiple-choice item g.

The Conditional Variances of Z_g Given the Realizations of U'_g

The variance of Z_g can assume only one of two possible values depending on whether item g was answered correctly or incorrectly. Expressions will now be derived for these variances given the two possible outcomes.

The Variance of Z_g Given a Correct Answer to Item g, $\sigma^2(Z_g \mid u'_g = 1)$. In order to solve for the variance of Z_g given a correct answer to item g, one first obtains an expression for $\mathscr{E}(Z_g^2 \mid u'_g = 1)$. This quantity represents the expected value for the sum of squared deviations from a mean that was appropriate before the truncation of Z_g at γ_g. One later obtains an expression for the desired variance, $\sigma^2 Z_g \mid u'_g = 1)$, or the expected value of the sum of squared deviations about $\mu(Z_g \mid u'_g = 1)$ as given in (3.3.21) through

$$\sigma^2(Z_g \mid u'_g = 1) = \mathscr{E}(Z_g^2 \mid u'_g = 1) - \mu^2(Z_g \mid u'_g = 1), \tag{3.3.32}$$

which provides an explicit solution for $\sigma^2(Z_g \mid u'_g = 1)$ given the identity

$$\mathscr{E}(Z_g^2 \mid u'_g = 1) = \sigma^2(Z_g \mid u'_g = 1) + \mu^2(Z_g \mid u'_g = 1). \tag{3.3.33}$$

Proceeding with the solution for $\mathscr{E}(Z_g^2 \mid u'_g = 1)$, it is known by definition that

$$\mathscr{E}(Z_g^2 \mid u'_g = 1) = \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g^2 \, \phi^{*\prime}(\zeta_g,\theta) \, d\theta \, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g^2 \, \phi^{*\prime}(\zeta_g,\theta) \, d\theta \, d\zeta_g, \tag{3.3.34}$$

where $\phi^{*\prime}(\zeta_g,\theta)$ is given by (3.3.12) and verified as the joint density function in the discussion surrounding (3.3.16). After a substitution from (3.3.12) into (3.3.34) one may write

$$\mathscr{E}(Z_g^2 \mid u'_g = 1) = \frac{1}{P'_g} \left[\int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g^2 \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g^2 \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g \right], \tag{3.3.35}$$

where (2.3.63) and (2.3.86) may be rearranged to yield

$$\mathscr{E}(Z_g^2 \mid u_g = 1)\, P_g = \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \zeta_g^2 \, \phi(\zeta_g,\theta) \, d\theta \, d\zeta_g \tag{3.3.36}$$

and

$$\mathscr{E}(Z_g^2 \mid u_g = 0)\, Q_g = \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g^2\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{3.3.37}$$

respectively, for substitution into (3.3.35). After these substitutions are made, the composited expected value may be expressed as

$$\mathscr{E}(Z_g^2 \mid u_g' = 1) = \left[\frac{P_g}{P_g'}\right] \mathscr{E}(Z_g^2 \mid u_g = 1) + \left[\frac{c_g Q_g}{P_g'}\right] \mathscr{E}(Z_g^2 \mid u_g = 0), \tag{3.3.38}$$

where the bracketed weights sum to unity because of (3.3.2) and provide the relative frequencies for the appropriate expected values that result from the dichotomization of Z_g. These expected values, of course, are identically those for the two-parameter normal ogive submodel. Substitutions from (2.3.81) and (2.3.95) into (3.3.38) provide

$$\mathscr{E}(Z_g^2 \mid u_g' = 1) = \frac{P_g}{P_g'}\left[1 + \frac{\gamma_g\, \phi(\gamma_g)}{P_g}\right] + \frac{c_g Q_g}{P_g'}\left[1 - \frac{\gamma_g\, \phi(\gamma_g)}{Q_g}\right], \tag{3.3.39}$$

which simplifies to

$$\mathscr{E}(Z_g^2 \mid u_g' = 1) = \frac{P_g + c_g Q_g}{P_g'} + (1 - c_g)\, \frac{\gamma_g\, \phi(\gamma_g)}{P_g'} \tag{3.3.40}$$

or, merely,

$$\mathscr{E}(Z_g^2 \mid u_g' = 1) = 1 + (1 - c_g)\, \frac{\gamma_g\, \phi(\gamma_g)}{P_g'}, \tag{3.3.41}$$

because of (3.3.2). Substitutions from (3.3.41) and from the squared result of (3.3.21) into (3.3.32), after some rearrangement, provide the desired result

$$\sigma^2(Z_g \mid u_g' = 1) = 1 - \frac{(1 - c_g)\, \phi(\gamma_g)}{P_g'}\left[\frac{(1 - c_g)\, \phi(\gamma_g)}{P_g'} - \gamma_g\right] \tag{3.3.42}$$

as a convenient expression for the variance of Z_g given a correct answer to a multiple-choice item g.

The Variance of Z_g Given an Incorrect Answer to Item g, $\sigma^2(Z_g \mid u_g' = 0)$. In order to solve for the variance of Z_g given an incorrect answer to a multiple-choice item g, $\sigma^2(Z_g \mid u_g' = 0)$, one first obtains an expression for $\mathscr{E}(Z_g^2 \mid u_g' = 0)$. This quantity represents the expected value for the sum of squared deviations from a mean that was appropriate before the truncation of Z_g at γ_g. One later obtains an expression for the desired variance, $\sigma^2(Z_g \mid u_g' = 0)$, or the expected value of the sum of squared deviations about $\mu(Z_g \mid u_g' = 0)$ as given in (3.3.31) through

$$\sigma^2(Z_g \mid u_g' = 0) = \mathscr{E}(Z_g^2 \mid u_g' = 0) - \mu^2(Z_g \mid u_g' = 0), \tag{3.3.43}$$

which provides an explicit solution for $\sigma^2(Z_g \mid u_g' = 0)$ given the identity

$$\mathscr{E}(Z_g^2 \mid u_g' = 0) = \sigma^2(Z_g \mid u_g' = 0) + \mu^2(Z_g \mid u_g' = 0). \tag{3.3.44}$$

Proceeding with the solution for $\mathscr{E}(Z_g^2 \mid u_g' = 0)$, it is known by definition that

$$\mathscr{E}(Z_g^2 \mid u_g' = 0) = (1 - c_g) \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g^2\, \phi^{\dagger\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{3.3.45}$$

where $\phi^{\dagger\prime}(\zeta_g,\theta)$ is given by (3.3.23) and verified as the joint density function for this particular probability integral in the discussion surrounding (3.3.27). A substitution from (3.3.23) into (3.3.45) allows one to write

$$\mathscr{E}(Z_g^2 \mid u_g' = 0) = \frac{(1 - c_g)}{Q_g'} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \zeta_g^2\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{3.3.46}$$

where a substitution from (3.3.37) yields

$$\mathscr{E}(Z_g^2 \mid u_g = 0) - \frac{(1 - c_g)\, Q_g}{Q_g'} \mathscr{E}(Z_g^2 \mid u_g = 0) \tag{3.3.47}$$

or, merely

$$\mathscr{E}(Z_g^2 \mid u_g' = 0) = \mathscr{E}(Z_g^2 \mid u_g = 0), \tag{3.3.48}$$

because of (3.3.7). Thus, the particular expected values given in (3.3.48) are identically defined for the two- and three-parameter normal ogive submodels. A substitution from (2.3.95) into (3.3.48) now yields

$$\mathscr{E}(Z_g^2 \mid u_g' = 0) = 1 - \frac{\gamma_g\, \phi(\gamma_g)}{Q_g}. \tag{3.3.49}$$

Final substitutions from (3.3.49) and from the squared result of (3.3.31) into (3.3.43), along with some rearrangement, now provide the desired result

$$\sigma^2(Z_g \mid u_g' = 0) = 1 - \frac{\phi(\gamma_g)}{Q_g} \left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g \right] \tag{3.3.50}$$

as a convenient expression for the variance of Z_g given an incorrect answer to a multiple-choice item g. Upon comparing (3.3.50) with (2.3.96), it may be noted that the variance of Z_g given an incorrect answer to item g is identically defined for the two- and three-parameter normal ogive submodels.

The expressions for the variances as provided in (3.3.42) and (3.3.50) will have further use in later derivations in Chapter 5. Specifically, these equations will provide inputs for the derivations of the restricted variances of ability given the realizations of the binary variable U_g'.

The Least Squares Estimators of Ability Given the Realizations of U_g'

Since the mean is the point about which the sum of squared deviations is minimized, least squares estimators of ability for the possible realizations of the binary variable U_g' are readily obtained.

(a) *Specific Case.* For the specific case where Θ has a mean of zero and a variance of unity, these estimators are merely the means of Θ given the two possible outcomes: the mean of Θ given a correct answer to a multiple-choice item g, $\mu(\Theta \mid u_g' = 1)$, and the mean of Θ given an incorrect answer to a multiple-choice item g, $\mu(\Theta \mid u_g' = 0)$. These least squares estimators of ability will now be derived for the specific case.

Specific Case: The Mean of Θ Given a Correct Answer to Item g, $\mu(\Theta \mid u_g' = 1)$. In order to obtain a convenient expression for the mean of Θ given a correct answer to a multiple-choice item g, one must evaluate the defining relationship

$$\begin{aligned}\mu(\Theta \mid u_g' = 1) &= \mathscr{E}(\Theta \mid u_g' = 1) \\ &= \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta\, \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta\, \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g,\end{aligned} \tag{3.3.51}$$

where $\phi^{*\prime}(\zeta_g,\theta)$ is given by (3.3.12) and verified as the joint density function in the discussion surrounding (3.3.16). A substitution from (3.3.12) into (3.3.51) allows the writing of

$$\begin{aligned}\mu(\Theta \mid u_g' = 1) &= \mathscr{E}(\Theta \mid u_g' = 1) \\ &= \frac{1}{P_g'} \left[\int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g \right].\end{aligned} \tag{3.3.52}$$

For subsequent substitution into (3.3.52), (2.3.98) and (2.3.110) may be rearranged to yield

$$\mu(\Theta \mid u_g = 1)\, P_g = \int_{\gamma_g}^{\infty} \int_{-\infty}^{\infty} \theta\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g \tag{3.3.53}$$

and

$$\mu(\Theta \mid u_g = 0)\, Q_g = \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{3.3.54}$$

respectively. After these substitutions are made, one may express the composited mean

$$\mu(\Theta \mid u_g' = 1) = \left[\frac{P_g}{P_g'}\right] \mu(\Theta \mid u_g = 1) + \left[\frac{c_g Q_g}{P_g'}\right] \mu(\Theta \mid u_g = 0), \tag{3.3.55}$$

resulting from the weighting of means. As can be deduced from (3.3.2), the bracketed weights that form the composite in (3.3.55) sum to unity. These weights thus represent the relative frequencies for the means that resulted from the dichotomization of Z_g at γ_g. These means, of course, are identically those defined for the two-parameter normal ogive submodel.

Substitutions from (2.3.107) and (2.3.115) into (3.3.55), along with some rearrangement, allow the writing of

$$\mu(\Theta \mid u_g' = 1) = \rho(Z_g,\Theta) \left\{\left[\frac{P_g}{P_g'}\right] \mu(Z_g \mid u_g = 1) + \left[\frac{c_g Q_g}{P_g'}\right] \mu(Z_g \mid u_g = 0)\right\}, \tag{3.3.56}$$

where a substitution from (3.3.20) provides

$$\mu(\Theta \mid u_g' = 1) = \rho(Z_g,\Theta)\, \mu(Z_g \mid u_g' = 1). \tag{3.3.57}$$

A final substitution from (3.3.21) into (3.3.57) yields the desired result

$$\mu(\Theta \mid u_g' = 1) = \rho(Z_g,\Theta)(1 - c_g) \frac{\phi(\gamma_g)}{P_g'} \tag{3.3.58}$$

as a convenient expression for the least squares estimator of Θ given a correct answer to a multiple-choice item g.

Specific Case: The Mean of Θ Given an Incorrect Answer to Item g, $\mu(\Theta \mid u_g' = 0)$. In order to obtain an expression for the mean of Θ given an incorrect answer to a multiple-choice item g, one must evaluate the defining relationship

$$\begin{aligned}\mu(\Theta \mid u_g' = 0) &= \mathscr{E}(\Theta \mid u_g' = 0) \\ &= (1 - c_g) \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta\, \phi^{\dagger\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g\end{aligned} \tag{3.3.59}$$

where $\phi^{\dagger\prime}(\zeta_g,\theta)$ is given by (3.3.23) and verified as the joint density function in the discussion surrounding (3.3.27). A substitution from (3.3.23) into (3.3.59) allows one to write

$$\begin{aligned}\mu(\Theta \mid u_g' = 0) &= \mathscr{E}(\Theta \mid u_g' = 0) \\ &= \frac{(1 - c_g)}{Q_g'} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g,\end{aligned} \tag{3.3.60}$$

where a substitution from (3.3.54) yields

$$\mu(\Theta \mid u_g' = 0) = \frac{(1 - c_g)\, Q_g}{Q_g'} \mu(\Theta \mid u_g = 0) \tag{3.3.61}$$

or, merely

$$\mu(\Theta \mid u_g' = 0) = \mu(\Theta \mid u_g = 0) \tag{3.3.62}$$

because of (3.3.7). Thus the least squares estimator of Θ given an incorrect answer to item g is identically defined in the submodels designed for free-response and multiple-choice items. This finding, of course, results from the assumption that success due to guessing is random on the interval of Z_g from negative infinity to γ_g. After a substitution from (2.3.115) into (3.3.62) one has

$$\mu(\Theta \mid u_g' = 0) = \rho(Z_g,\Theta)\, \mu(Z_g \mid u_g = 0), \tag{3.3.63}$$

where a final substitution from (2.3.59) yields the desired result

$$\mu(\Theta \mid u_g' = 0) = -\rho(Z_g,\Theta) \frac{\phi(\gamma_g)}{Q_g} \tag{3.3.64}$$

as a convenient expression for the least squares estimator of Θ given an incorrect answer to a multiple-choice item g.

General Case. As mentioned earlier, the general case allows the examiner to prescribe the scale of ability or the mean and variance of Θ^*, $\mu(\Theta^*)$, and $\sigma^2(\Theta^*)$. Under this case Θ may be viewed as resulting from the standardization of Θ^*. This relationship was given in (2.3.117). An explicit solution for θ^* was given in (2.3.118). It was also mentioned that the general case subsumed the specific case of least squares estimators when $\mu(\theta^*)$ was prescribed as zero and $\sigma^2(\Theta^*)$ was prescribed as unity. Further it was noted that the item parameters a_g and b_g of (2.1.12) and (2.1.13), respectively, are not appropriate for the general case—the sole exception being the subsumed case. The appropriate parameters, a_g^* and b_g^* were defined in (2.3.119) and in (2.3.120), respectively. These parameters are still appropriate for a multiple-choice item g or the three-parameter normal ogive submodel given the general case. In the instance of the third parameter, c_g^*, one merely has

$$c_g^* = c_g, \tag{3.3.65}$$

where c_g^* represents the lower asymptote of the regression of a multiple-choice item g on Θ^*. These item parameters guarantee that the probability of obtaining a correct answer to a multiple-choice item g remains invariant under arbitrary prescriptions for the scale of ability, that is, the mean and variance of Θ^*, $\mu(\Theta^*)$, and $\sigma^2(\Theta^*)$. Convenient expressions for $\rho(Z_g,\Theta)$ given a_g^* and γ_g given a_g^* and b_g^* were provided earlier in (2.3.126) and (2.3.128), respectively.

For the general case, the least squares estimators of ability are merely the means of Θ^* given the two possible outcomes: the mean of Θ^* given a correct answer to a multiple choice item g, $\mu(\Theta \mid u_g' = 1)$, and the mean of Θ^* given an incorrect answer to a multiple-choice item g, $\mu(\Theta^* \mid u_g' = 0)$. Estimators of ability for this general case will now be derived.

General Case: The Mean of Θ^* Given a Correct Answer to Item g, $\mu(\Theta^* \mid u_g' = 1)$. In order to obtain an expression for the mean of Θ^* given a correct answer to a multiple-choice item g, one must evaluate the defining relationship

$$\begin{aligned}\mu(\Theta^* \mid u_g' = 1) &= \mathscr{E}(\Theta^* \mid u_g' = 1) \\ &= \int_{\gamma_g}^{\infty}\int_{-\infty}^{\infty} \theta^* \, \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \theta^* \, \phi^{*\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g,\end{aligned} \tag{3.3.66}$$

where $\phi^{*\prime}(\zeta_g,\theta)$ is given by (3.3.12), verified as the joint density function in the discussion surrounding (3.3.16), and still appropriate here because of (2.3.117) and (2.3.118). A substitution from (3.3.12) into (3.3.66) allows the writing of

$$\begin{aligned}\mu(\Theta^* \mid u_g' = 1) &= \mathscr{E}(\Theta^* \mid u_g' = 1) \\ &= \frac{1}{P_g'}\left[\int_{\gamma_g}^{\infty}\int_{-\infty}^{\infty} \theta^* \, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g + c_g \int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \theta^* \, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g\right],\end{aligned} \tag{3.3.67}$$

where (2.3.130) and (2.3.141) may be rearranged to yield

$$\mu(\Theta^* \mid u_g = 1)\, P_g = \int_{\gamma_g}^{\infty}\int_{-\infty}^{\infty} \theta^* \, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g \tag{3.3.68}$$

and

$$\mu(\Theta^* \mid u_g = 0)\, Q_g = \int_{-\infty}^{\gamma_g}\int_{-\infty}^{\infty} \theta^* \, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g, \tag{3.3.69}$$

respectively, for substitution into (3.3.67). After these substitutions are made, one may express the composited mean

$$\mu(\Theta^* \mid u_g' = 1) = \left[\frac{P_g}{P_g'}\right] \mu(\Theta^* \mid u_g = 1) + \left[\frac{c_g Q_g}{P_g'}\right] \mu(\Theta^* \mid u_g = 0), \tag{3.3.70}$$

resulting from a weighting of means. As can be deduced from (3.3.2), the bracketed weights that form the composite sum to unity. These weights thus represent the relative frequencies for the means that resulted from the dichotomization of Z_g at γ_g. These weighted means, of course, are identically those defined for the two-parameter normal ogive submodel.

Substitutions from (2.3.138) and (2.3.146) into (3.3.70), along with some rearrangement, allow one to write

$$\mu(\Theta^* \mid u'_g = 1) = \left[\frac{P_g}{P'_g}\right] [\mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, \mu(Z_g \mid u_g = 1)]$$
$$+ \left[\frac{c_g Q_g}{P'_g}\right] [\mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, \mu(Z_g \mid u_g = 0)]. \quad (3.3.71)$$

Because of (3.3.2) one may write

$$\mu(\Theta^* \mid u'_g = 1) = \mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*) \left\{ \left[\frac{P_g}{P'_g}\right] \mu(Z_g \mid u_g = 1) + \left[\frac{c_g Q_g}{P'_g}\right] \mu(Z_g \mid u_g = 0) \right\} \quad (3.3.72)$$

where a substitution from (3.3.20) into (3.3.72) provides

$$\mu(\Theta^* \mid u'_g = 1) = \mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, \mu(Z_g \mid u'_g = 1). \quad (3.3.73)$$

A final substitution from (3.3.21) into (3.3.73) yields the desired result

$$\mu(\Theta^* \mid u'_g = 1) = \mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)(1 - c_g) \frac{\phi(\gamma_g)}{P'_g} \quad (3.3.74)$$

as a convenient expression for the least squares estimator of ability, Θ^*, given a correct answer to a multiple-choice item g.

General Case: The Mean of Θ^* Given an Incorrect Answer to Item g, $\mu(\Theta^* \mid u'_g = 0)$. In order to obtain an expression for the mean of Θ^* given an incorrect answer to a multiple-choice item g, $\mu(\Theta^* \mid u'_g = 0)$, one must evaluate the defining relationship

$$\mu(\Theta^* \mid u'_g = 0) = \mathscr{E}(\Theta^* \mid u'_g = 0)$$
$$= (1 - c_g) \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta^*\, \phi^{\dagger\prime}(\zeta_g,\theta)\, d\theta\, d\zeta_g, \quad (3.3.75)$$

where $\phi^{\dagger\prime}(\zeta_g,\theta)$ is given by (3.3.23), verified as the joint density function in the discussion surrounding (3.3.27), and still appropriate because of (2.3.117) and (2.3.118). A substitution from (3.3.23) into (3.3.75) allows the writing of

$$\mu(\Theta^* \mid u'_g = 0) = \mathscr{E}(\Theta^* \mid u'_g = 0)$$
$$= \frac{(1 - c_g)}{Q'_g} \int_{-\infty}^{\gamma_g} \int_{-\infty}^{\infty} \theta^*\, \phi(\zeta_g,\theta)\, d\theta\, d\zeta_g, \quad (3.3.76)$$

where a substitution from (3.3.69) yields

$$\mu(\Theta^* \mid u'_g = 0) = \frac{(1 - c_g)\, Q_g}{Q'_g}\, \mu(\Theta^* \mid u_g = 0) \quad (3.3.77)$$

or, merely

$$\mu(\Theta^* \mid u'_g = 0) = \mu(\Theta^* \mid u_g = 0), \quad (3.3.78)$$

because of (3.3.7). Thus the least squares estimator of Θ^* given an incorrect answer to item g is identically defined for the submodels designed for the free-response and multiple-choice cases. The finding, of course, is a result of the assumption that success due to guessing in the case of the three-parameter normal ogive submodel is assumed to occur at random on the interval of Z_g from negative infinity to γ_g. After a substitution from (2.3.146) into (3.3.78), one has

$$\mu(\Theta^* \mid u'_g = 0) = \mu(\Theta^*) + \rho(Z_g,\Theta)\, \sigma(\Theta^*)\, \mu(Z_g \mid u_g = 0), \quad (3.3.79)$$

where a final substitution from (2.3.59) into (3.3.79) yields the desired result

$$\mu(\Theta^* \mid u'_g = 0) = \mu(\Theta^*) - \rho(Z_g,\Theta)\, \sigma(\Theta^*) \frac{\phi(\gamma_g)}{Q_g} \quad (3.3.80)$$

as a convenient expression for the least squares estimator of Θ^* given an incorrect answer to a multiple-choice item g.

The expressions for the least squares estimators provided by (3.3.73), (3.3.74), (3.3.79), and (3.3.80) will have further use in later derivations in Chapter 5. Notice that the estimators provided by (3.3.58) and (3.3.64) may be viewed as resulting from (3.3.74) and (3.3.80), respectively, under the simplifying assumption that $\mu(\Theta^*)$ is zero and $\sigma^2(\Theta^*)$ is unity. In Chapters 5 and 6, the primary concern will be with estimation under the general case, in particular, the sequential estimation of Θ^* given the items that are chosen to constitute a tailored test. Algorithms will be developed for tailored testing with both free-response and multiple-choice items. But before these developments occur, a general discussion of the tailored testing process is in order.

4. THE TAILORED TESTING PROCESS

In the preceding two chapters, two normal ogive submodels were derived from the model for a single common factor. These submodels form the mathematical basis for the process of tailored testing in those particular instances where the basic assumptions are reasonable for an adequate number of items. The two-parameter submodel should be selected as the basis for this process when a sufficient supply of free-response items that are scored as either correct or incorrect is available. When a sufficient supply of multiple-choice items that are scored as either correct or incorrect is available, the three-parameter submodel should be selected as the basis for this process. In the following two chapters, tailoring algorithms for each of these submodels will be derived with the aid of selection theory. These algorithms will represent the explicit mathematical statements basic to the tailoring process. In the present chapter, however, concern will focus on the less specific: Informal tests of the basic assumptions or the validity of these submodels will be discussed; the requirements of the tailoring process will be delineated; and the tailoring of an individual test will be described.

4.1 The Validity or Adequacy of Fit of the Normal Ogive Submodels

In assessing the validity of submodels such as these, interest centers on determining the extent of agreement between the particular submodel and the observed data. The extent of agreement should be adequate. For instance, it can usually be assumed that given large enough samples such submodels will show some departure from the observed data. What is required then is a degree of agreement sufficient to produce useful results upon the application of the particular submodel. That is to say, the submodel should have some practical value. Major discrepancies between the submodel and observed data would ordinarily be invalidating. Such discrepancies should severely limit the usefulness of produced results. On the other hand, the practical implications of minor discrepancies might be nonexistent.

In regard to testing the adequacy of fit of statistical models, Birnbaum (1968) has offered the following observations:

> Where specific techniques of testing fit are concerned, the reader should be aware that some established approaches to testing goodness of fit have come to be considered unsound and potentially misleading by a number of statisticians and scientific workers. An alternative perspective on testing adequacy of models is one based primarily on rather direct, often graphical, comparisons of data with significant aspects of models. Here a crucial role is played by relatively unformalized judgments . . . . (pp. 422–423)

There are several basic assumptions underlying the submodels that have been developed. The reasonableness of these basic assumptions should be assessed before a major application of tailored testing is initiated. One basic assumption is that of unidimensionality. The items must form a unifactorial or unidimensional set. This implies that the item error scores are unique or independent from item to item. As a result, the item covariances should vanish when ability is partialled out as indicated in (1.1.14). Another basic assumption, also implied in the basic model for a single common factor, is that of local independence. This assumption requires that the item covariances vanish when ability is held locally constant as well as when ability is completely partialled out. A further basic assumption is that ability is normally distributed in the population of interest. A normal distribution of ability is assumed in the model for a single common factor. A final basic assumption is that the regressions of the binary items U_g or U_g' on ability are two- or three-parameter normal ogives depending on item type. As delineated in Chapters 2 and 3, this is a direct consequence of the assumed bivariate normality of the Z_g and Θ as well as the determining relationship between the Z_g and the U_g and the Z_g and the U_g'.

The Separate Evaluations of Basic Assumptions

One method of testing the validity of a particular submodel might be to consider its basic assumptions as aspects of significance and separately assess the reasonableness of each. While this method might be easily applied in the instance of some basic assumptions, its application to others may be extremely difficult, if not impossible, and some practices in assessing the reasonableness of the several basic assumptions have been both unsound and misleading.

Unidimensionality. The usual approach, in assessing whether the assumption of unidimensionality is reasonable, is to factor analyze the set of items. If a single common factor provides an adequate description of the observed data, that is, if the sample analogue of the matrix $\Sigma(\Delta,\Delta)$ of (1.1.14) is essentially diagonal, then the assumption is deemed reasonable. This approach appears deceptively straightforward, but it is fraught with potential for misapplication. The approach is only appropriate when free-response items are being analyzed. In this case, the item intercorrelations must be tetrachoric coefficients. The correlation coefficient of (1.1.21) can be estimated by a tetrachoric coefficient, given data from the binary variables U_g of free-response items. If phi coefficients are used for this purpose a proliferation of "difficulty" factors can occur even when the state of nature is in consonance with the unidimensional assumption. Phi coefficients are influenced by item difficulties. When the state of nature is unidimensional, sets of items of similar difficulty can produce a spurious multidimensionality. The approach should not be used with multiple-choice items even with tetrachoric coefficients. The tetrachoric coefficient assumes that the binary variables arising from a score of *one* if correct and a score of *zero* if incorrect result from the dichotomization of normal random variables such as the Z_g. While this assumption is reasonable for free-response items, it is clearly unreasonable for any pair of multiple-choice items. Under the submodel designed for multiple-choice items, the binary scores of *one* and *zero* are not fully determined by dichotomizations of the continuous variables Z_g. A brief review of Figure 3.1.1 should convince the reader of the unreasonableness of this assumption in the multiple-choice case when the three-parameter normal ogive submodel has been found to be valid. Given data from binary variables such as the U'_g, the tetrachoric coefficient is clearly not an estimate of the correlation coefficient of (1.1.21). A spurious multidimensionality can result from the misapplication of factor analysis to multiple-choice items. This improper "tetrachoric coefficient" is also influenced by item difficulties.

In the case of multiple-choice items, the advocated approach is to select items with high values of a_g. High values of a_g indicate strong relationships between the items and the particular dimension of ability. Notice in (2.2.10) that a_g completely determines $\rho(Z_g,\Theta)$, and that this particular correlation coefficient, in the usual factor-analytic parlance, is a factor loading. When the particular dimension of ability has a long and well-established factor-analytic pedigree, the values of a_g tend to be rather high. In fact, these values tend to be much higher than previously thought. Selecting items with high values of a_g then tends toward a unidimensional set for the well-established dimensions of ability. Remember, it is not essential that all items encountered be adequately represented by a particular submodel. Adequate representation is only necessary for those items selected for use in the tailoring process. In Section 4.2, more will be said about item selection.

Local Independence. Local independence is a higher order of statistical independence than indicated by matrix $\Sigma(\Delta,\Delta)$ of (1.1.14), and is a state of nature where the item variance-covariance matrices are diagonal matrices for arbitrary values of ability. That is to say, the item covariances vanish when ability is held constant. To separately assess the reasonableness of this assumption, therefore, would be extremely difficult if not impossible. In latent trait applications, one is usually limited to finite samples of examinees as well as to estimates of ability. A difficult problem, then, is the size of subsamples when the estimate of ability is held constant. Another difficult problem is that true ability is not fixed when estimates of ability are held constant. Thus, the appropriate variance-covariance matrices for use in assessing the reasonableness of this assumption are usually not obtainable in practice. That the sample analogue of the matrix $\Sigma(\Delta,\Delta)$ of (1.1.14) be essentially diagonal is a necessary condition given the reasonableness of the assumption of local independence. However, this condition in and of itself is not sufficient to establish the reasonableness of this basic assumption.

Normally Distributed Ability. The assumption of normally distributed ability is almost always reasonable. In fact, it is difficult to construct tailored tests under the procedures advocated in this report that will be inconsistent with this assumption. In Table 5.3.1, for example, data will be presented which indicate that the assumption of normality provides a very workable approximation even when the distributions of ability are known to be nonnormal.

If the distributions of ability estimates are examined to assess normality, the nature of the estimates must be kept in mind. When ability is estimated by the algorithms of Section 6.2 and 6.3, the estimates are approximate least squares or regressed estimates. One should ensure that these estimates can be expected to form a linear regression on the Z_g. Only under this condition will the form of the distribution of ability estimates reflect that of the distribution of true ability. This condition is ensured by terminating each tailored test when the error of the ability estimate is at most some constant. Given a constant terminal error, the mean and variance of the distribution of ability estimates can also be predicted. A detailed consideration will be given to the termination of tailored tests in Chapter 7.

Normal Ogive Regressions. Given these submodels, normal ogive regressions of binary items on estimates of ability can be expected when the terminal error is held constant for each tailored test. If the distribution of estimates of ability departs significantly from the distribution of true ability, these regressions can also depart significantly from their true form. In this context, tests of goodness of fit should be considered unsound and potentially misleading, for two reasons. Normal ogive regressions are not expected when the distributional forms of estimates of ability and true ability depart significantly; and minor discrepancies from assumed regressions may have insignificant practical implications. Minor discrepancies are easily detected given large enough samples.

In order to assess the reasonableness of this basic assumption for a particular submodel, graphs of the sample analogues of these regressions can be visually inspected. The conditions under which the true forms of these regressions are to be expected should be kept in mind. The graphs for free-response items should be represented reasonably well by two-parameter normal ogives; whereas three-parameter normal ogives should represent these graphs reasonably well for multiple-choice items.

The Alternative Perspective

Another method of testing the validity of a particular submodel is to consider significant aspects of practical value. In this case, results of practical importance are predicted. These predictions are derived or inferred from the submodel. Thus, they depend on the reasonableness of all of the basic assumptions. The predictions are then subjected to disconfirmation using observed data. If the predicted results are obtained, the basic assumptions are considered reasonable. If the predicted results are disconfirmed, then the basic assumptions are considered unreasonable. Normally, this method is applied only after several of the basic assumptions have been assessed as reasonable.

An example of the application of this method has been reported in the literature (Urry, 1977, pp. 188–190). Accepting all of the basic assumptions of the three-parameter submodel as the true state of nature, the investigator predicted some eight correlations between distinct measures of the same ability. These predictions were for correlations between eight ability estimates resulting from tailored testing to eight different levels of reliability and the test score from a written test of known reliability. A pilot study of tailored testing was then conducted. In each of the eight cases, the obtained correlation exceeded in numerical value the corresponding predicted correlation. As a consequence, it may be concluded that the three-parameter normal ogive submodel, given all of its basic assumptions, is valid for use with multiple-choice items. In Chapter 7, the theoretical basis for the prediction of these correlations will be derived and discussed.

Aside from the important implication for validity, a practical implication drawn in the cited literature for tailored testing with this submodel was that the levels of predicted reliability could be achieved with dramatic reductions in the number of required items *vis-a-vis* conventional paper-and-pencil testing. The observed reductions were of the order of 80 percent. The practical importance of these reductions is principally economic. Fewer tailored testing stations, at a given cost, are required to sustain an established testing volume. Given this established volume, the amount of use of testing stations, and therefore their number, depends primarily on testing time, which bears a direct relationship to test length.

4.2. Requirements of the Tailored Testing Process

Given the validity or adequacy of fit of a particular submodel, a successful application of the tailoring process depends on (a) accurate estimates of the item parameters and (b) an appropriate selection of items for the ability bank.

Accurate Estimates of the Item Parameters

In the developments thus far it has been assumed that the item parameters—either a_g and b_g in the case of the two-parameter normal ogive submodel or a_g, b_g, and c_g in the case of the three-parameter normal ogive submodel—are known. In practice, these parameters need to be estimated. Accuracy in estimating the item parameters depends on both the number of items and the number of examinees in the analyzed data sets. If applications of tailored testing are anticipated, one should ensure that adequate data are available for this purpose.

Algorithms for the estimation of item parameters will be presented in a subsequent report. The basic assumptions of these algorithms are identical to those underlying the normal ogive submodels discussed here. Simulation techniques will be considered that will allow an evaluation of the accuracy of item parameter estimates.

Notice that the expressions derived for the submodels in Chapters 2 and 3, as well as those yet to be derived, all assume or will assume that the item parameters are known. Predictions drawn from these submodels, then, also assume knowledge of the item parameters; and predictions of this type will become an integral part of the tailoring process. Thus, accurate estimation or reasonable knowledge of the item parameters is essential.

An Appropriate Selection of Items for the Ability Bank

The item parameters are usually estimated with ability scaled to a mean of zero and a variance of unity. Thus one usually has estimates of a_g, b_g, and, in the case of the three-parameter normal ogive submodel, c_g of equations (2.1.12), (2.1.13),and (3.1.17), respectively. Item selection is best made on the basis of these estimates. Later, these estimates may be transformed to a_g^*, b_g^*, and, in the case of the three-parameter normal ogive submodel, c_g^*, by means of equations (2.2.17), (2.2.18),and (3.2.14), respectively, when a mean and variance have been prescribed for the scale of ability.

In item selection, there are three primary goals: (a) to construct an item bank where the constituent items are adequately represented by a particular submodel and consequently measure a single dimension of ability—hence the designation *ability bank*; (b) to obtain constituent items for the ability bank such that their required numbers in achieving given levels of reliability through tailored testing is reduced relative to the item requirements of conventional paper-and-pencil testing; and (c) to obtain constituent items for the ability bank such that the estimates of ability obtained for all examinees can be essentially equiprecise, that is, the error in ability estimation at the termination of tailored testing can be essentially constant for all examinees.

In Section 4.1, several methods of evaluating the adequacy of fit of these submodels to item data were discussed. To obtain the first goal in item selection, then, the methods prescribed in that section should be applied. The application of the unprescribed methods, that are known to be unsound and potentially misleading, should be avoided.

To obtain the second goal, one needs to select items with high values of a_g and, in the case of the three-parameter normal ogive submodel, low values of c_g. In both cases, the b_g must be evenly and widely distributed.

It has been demonstrated (Urry, 1970, 1977) that a minimum value of .8 should be set for the a_g. This minimum value ensures that tailored testing will be more economical in item usage than conventional testing in achieving the same level of reliability. Higher values of a_g improve this economic outlook. When the regressions of the binary items on ability are essentially linear, the test score of conventional testing, the number of correct items, cannot be improved on in this regard. It is only when these regressions become markedly curvilinear, or when a_g exceeds .8 in value, that dramatic reductions in the number of required items can be realized through tailored testing. Under the two-parameter normal ogive submodel, the biserial correlation between a free-response item and a perfectly reliable conventional test is about .62 when a_g is .8, irrespective of the item's difficulty. However, given the three-parameter normal ogive submodel and a_g equal to .8, the biserial correlation between a multiple-choice item and a perfectly reliable conventional test varies with the item's difficulty as well as with c_g, the item's coefficient of guessing. For any nonzero c_g, the value of this biserial correlation is less for difficult items than for easy items. Since success due to guessing occurs with greater frequency on difficult items than on easy items, its masking effect on item discriminatory power as indexed by the biserial correlation is more pronounced for difficult items than for easy items. When a_g is .8 and c_g is .20, this biserial correlation is about .26 for very difficult items and about .59 for very easy items (Urry, 1976). Since this biserial correlation is also attenuated by unreliability in the conventional test, experience with the biserial correlation in conventional item analysis should lead one to conclude that obtaining multiple-choice items where a_g is at least .8 is a far easier task than is commonly thought.

High values of c_g have adverse effects on the tailored testing process. The gain per item in precision is lessened in sequential ability estimation. Thus, multiple-choice items having more rather than fewer alternatives are desirable. As a rule of thumb, a maximum value of .3 is suggested for the c_g, where c_g is somewhat related to, but not fixed by the number of item alternatives.

The item difficulties, the b_g, should be distributed in a fashion that will allow redundant challenge at all levels of ability in the population of interest. Redundancy is required in order to obtain equal precision in ability estimation for the full range of ability in this population. The distribution of b_g, then, should be evenly and widely distributed. How evenly and widely distributed depends on the degree of precision or reliability required of ability estimates, which brings one to the third goal.

Consideration of the third goal requires a clearer definition of reliability in the tailored testing context. The process of tailored testing produces as many tests as there are examinees. Each test is tailored to an examinee of given ability. In this context, reliability is a property of the estimates of ability resulting from the process. In the conventional context of paper-and-pencil testing, reliability is, by way of contrast, a property of a specific printed form of the test.

An index of reliability for the estimates of ability resulting from the tailoring process will be derived in Chapter 7. As will be seen, this index represents a binary analogue of the squared multiple correlation. It possesses the following similarities to reliability in classical test theory: (a) this index represents the correlation between equiprecise estimates of the same ability which is analogous to the correlation between parallel forms in the classical setting, and (b) the square root of this index represents the correlation between equiprecise ability estimates and true ability, which is analogous to the correlation between test scores and true score.

The use of this index assumes constant conditional variances of Θ^* given the tailored tests. As mentioned earlier, this constancy ensures that the estimates of ability form a linear regression. A linear regression can be approximated by terminating each tailored test when the error of the ability estimate is equal to or less than a prespecified constant. Given this prespecified constant, a lower bound to this index of reliability can be calculated. This calculation is possible because appropriate termination guarantees that the error in ability estimation is at most equal to the prespecified constant. As the prespecified constant becomes smaller, or as the achieved termination rule becomes more stringent, the error in ability estimation more closely approaches this prespecified constant. The smaller the prespecified constant is set, the longer each tailored test must be in order to achieve the termination rule; and the change per item in the error of ability estimation decreases with tailored test length. Thus the longer the tailored test, the more closely the error in ability estimation approaches the prespecified constant. As the expected value of the error in ability estimation, taken over the tailored tests, approaches the prespecified constant, tailored test reliability approaches the lower bound to the index of reliability. The lower bound to the index of reliability will be discussed in Chapter 7. Its calculation allows a forecast of tailored test reliabilities before the process of tailored testing begins.

There is an important condition. The prespecified constant must be reached at termination in each tailored test. Achieving this condition depends on the value of the prespecified constant and on the configuration of item parameters in the ability bank. If higher levels of precision or reliability are necessary, closer spacing is required between the item difficulties. In turn, this requirement calls for an increase in the number of items in the ability bank. In order to determine the performance of the ability bank in regard to achieving desired tailored test reliabilities, simulation studies (Urry, 1974, 1976, 1977) should be conducted. For a set of multiple-choice items, such simulation studies consist of applying the tailored testing algorithm presented in Section 6.2 for the three-parameter normal ogive submodel to binary item data that has been generated from this submodel. The generation of binary item data given this submodel will be discussed in Section 11.1. But briefly, in this generation process, the estimates of the parameters a_g, b_g, and c_g for the items in a particular ability bank are used. As a part of this process, values of true ability are sampled from the distribution assumed for Θ. Thus, the values of true ability are known. Estimates of ability can then be obtained through tailored testing with the algorithm given data for items with the parametric configuration of this particular ability bank and the assumed distribution of ability. The estimates of ability obtained under certain termination rules can then be correlated with the known values of the ability. Since the squared correlation between the estimates of ability and the known values of ability defines tailored test reliability, one can determine how well tailored test reliability is forecasted from the termination rules as well as how stringent the termination rule can be made, such that the prespecified constant can still be achieved in the tailored testing of individuals with this specific ability bank. Simulation studies such as these allow an assessment as to the achievability of tailored test reliabilities given the parametric configuration of a specific ability bank.

Subject to revision through the interpretation of results from simulation studies, it is suggested that item difficulties be evenly distributed on the interval from -2.0 to 2.0, and that the total number of items be approximately 100. Given the ability estimation techniques provided in this report, a wider distribution of item difficulties and a greater number of items are required when higher levels of reliability, say those in excess of .95, are deemed necessary.

4.3 Tailoring the Individual Test

In the present section general features of the tailoring process will be described. In Chapters 5 and 6, the formal mathematical statements or basic algorithms for this process will be derived for both the two- and three-parameter normal ogive submodels. As will be seen later, tailored testing is best implemented as a computer-interactive process because of a requirement for extensive mathematical calculation. This requirement will become apparent in Chapter 8, where a numerical illustration of an individually tailored test is provided.

From the perspective of the individual, the tailoring process involves the following: the individual is seated in front of a microcomputer. Instructions on the proper use of this equipment are provided on its cathode ray tube, or television-like screen. The instructions elicit series of actions. When the elicited actions indicate the individual's complete understanding of what this situation requires, the test begins. A test item appears on the cathode ray tube, and the individual provides an answer on the typewriter-like keyboard of this microcomputer. If the answer given is correct, the next item will be more difficult. If the answer given is not correct, an easier item will follow. The process continues in similar fashion until a message on the cathode ray tube indicates that testing has been concluded.

The overall goal of tailored testing is to obtain equally precise estimates of ability. In obtaining this goal, the process has a scenario that is constant from individual to individual. The general scenario unfolds as follows: An individual known to be a member of the population of interest is to be tested. Given what is known about this individual, the least-squares estimate of ability is the mean of ability, $\mu(\Theta^*)$, and its squared error, the variance of ability, is $\sigma^2(\Theta^*)$. These prior estimates, $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$, are then used to initiate the following iterated sequence:

1. Given the current estimate of ability and its variance, the most informative item is chosen from the ability bank;
2. This item is presented on the cathode ray tube;
3. The individual responds by way of the keyboard;
4. The response is scored;
5. A revised estimate of ability is obtained;
6. Its variance is estimated;
7. The square root of this variance, the error of the estimate of ability, is compared to a prespecified constant;
8. If the error of the ability estimate is equal to or less than the prespecified constant, the test is terminated;
9. Otherwise, the sequence is repeated.

The sequence is repeated until the condition in Step 8 is satisfied. Satisfaction of Step 8 has important implications with respect to the linearity of regression of the ability estimates and the reliability of the tailored testing process. The development of the algebraic intricacies in the applications of this general scenario requires the further derivation provided in the following two chapters.

For this later development, it is important to distinguish two necessary systems of item subscription. The items in the ability bank are subscripted g, where $g = 1, 2, \ldots$ p. These items are then chosen from the ability bank as in Step 1 to form the individualized sequence that becomes the tailored test. Within a tailored test, the items chosen from the ability bank are subscripted in sequential order by n, where $n = 1, 2, \ldots q_i$. In this context, q_i is the number of repetitions of the iterated sequence for individual i. The number of items required to satisfy Step 8, thus terminating the iterations, can vary with individuals. Variability in tailored test length is necessary to insure that the degree of precision imposed by Step 8 is constant across individuals. For instance, fewer items are required for individuals of average ability than are required for individuals of high ability to achieve termination in Step 8.

5. PRELIMINARY DEVELOPMENTS FOR THE TAILORED TESTING ALGORITHMS

5.1. The Selection or Rejection Analogy

In this chapter, selection theory is used as a basis for the development of the tailored testing algorithms. The relevance of this theory can be established by analogy. In Chapters 2 and 3, the two- and three-parameter normal ogive submodels were distinguished from one another on the basis of the hypothetical relationships between the continuous random variables Z_g and the binary random variables U_g or U'_g. As portrayed in Figure 2.1.1 and Figure 3.1.1, the Z_g determine the U_g and U'_g. The manner of this determination is the characteristic distinguishing between these submodels. These hypothetical relationships are now viewed from a selection perspective. One should remember that the Z_g are not observed, although as described in Chapters 2 and 3, they might be. The observed U_g and U'_g, derive from the unobserved Z_g. In tailored testing, one might view the process as though binary scores, which are the realizations of U_g or U'_g, indicate explicit selection on the continuous variable Z_g. That is to say, binary scoring is analogous to explicit selection or explicit rejection on a specific continuous variable Z_g. The similarity is not complete because the "cut score", γ_g, cannot be changed on a particular Z_g. The approach, then, has to be one of identifying and choosing a particular item g that has a desirable "cut score," γ_g. This feature of the submodels introduces the requirement for a sufficient supply of items in order to allow adequate flexibility in choosing the "cut scores," the γ_g.

Under the analogy, the correspondence between the binary scoring of an item and explicit selection or rejection on the continuous variable Z_g can be delineated as well as illustrated. For the two-parameter submodel, the realization u_g equal to *one* is the analogue of an explicit selection on Z_g of that subpopulation with a free-response score, ζ_{gio}, equal to or greater than a "cut score," γ_g. Similarly, the realization u_g equal to *zero* is the analogue of explicit rejection on Z_g of that subpopulation with free-response scores, ζ_{gio}, less than a "cut score," γ_g. These analogous relationships may be reviewed by consulting the shaded and unshaded areas of Z_g in Figure 2.1.1.

Explicit selection and explicit rejection on Z_g as indicated by the realization of U'_g for the three-parameter submodel, under the same analogy, are somewhat more complicated. The realization u'_g equal to *one* is the analogue of explicit selection on Z_g of that subpopulation with recognition scores, ζ_{gio}, equal to or greater than a "cut score," γ_g, plus a random proportion c_g of that subpopulation with recognition scores, ζ_{gio}, less than a "cut score," γ_g. This method of explicit selection is indicated by the shaded area of Z_g as presented in Figure 3.1.1. Similarly, the realization u'_g equal to *zero* is the analogue of explicit rejection on Z_g of the remaining proportion, $(1 - c_g)$, of that subpopulation with recognition scores, ζ_{gio}, less than a "cut score," γ_g. This method of explicit rejection is graphically indicated by the unshaded area of Z_g as presented in Figure 3.1.1. These analogous relationships support the relevance of selection theory for further developments.

It was mentioned earlier that items are sequentially chosen from the ability bank to construct each tailored test. In sequence, each choice obtains the most informative, available item g given the current information on the individual. From this perspective, a tailored test is analogous to a series of explicit selections and explicit rejections indicated by an outcome vector containing the individual's binary scores on the most informative items from the ability bank. The mean ability score for the subpopulation that would have been obtained given the series of explicit selections and explicit rejections is taken as the estimator of the individual's ability. Under the analogy, the variance of this estimator is the restricted variance of ability for the subpopulation that would have been obtained by the associated series of incidental selections and incidental rejections on the variable of ability as indicated by the outcome vector of binary scores. This variance is the variance of ability given restriction in range, the effect of the incidental selections and rejections. In this context, restriction in range is desirable. It denotes precision in ability estimation.

Viewed from the perspective of the analogy, the sequential choice of items produces a series of explicit selections and rejections resulting in the identification of a subpopulation that is relatively homogeneous with respect to the variable of ability. That is to say, items are sequentially chosen in order to restrict most severely the variance and thus the range of ability associated with a particular outcome vector of binary scores indicating explicit selections and rejections on a series of Z_n.

In Section 5.2, a general expression will be developed for the choice of the most informative items from the ability bank. It will be established that the chosen items are the most informative items to use in the construction of the particular tailored test. In Section 5.3, some important developments are derived from selection theory. These developments will provide the needed input for the recursive formulation of the tailoring algorithms to be presented in Chapter 6. In Sections 6.2 and 6.3, the tailoring algorithms for the two- and three-parameter normal ogive submodels will be summarized from earlier developments, mainly those of Section 5.3.

In the following discussion, it is assumed that "incidental selection" and "incidental rejection" with respect to the variable of ability result in a normal distribution of ability. This assumption represents a very workable approximation allowing the sequential estimation of ability and its variance. Sequential estimation under this assumption is generally referred to as a restricted updating procedure.

5.2 The Sequential Choice of Items

In this context, items are chosen by the tailoring algorithm. Item selection, as discussed in Section 4.2, consisted of the choosing of items by the practitioner. This item selection produces an effective ability bank of manageable size. Since items are later chosen by the algorithm, one can usually further reduce the size of the ability bank. This reduction can be accomplished by eliminating those items never chosen by the algorithm when the population of interest is being tested to the required level of reliability. This process of elimination does not influence the effectiveness of the ability bank, but renders its size, that is, its number of items, more manageable in a microcomputer environment.

It was mentioned earlier that the goal of tailored testing was to obtain equally precise estimates of ability. Within the individually tailored test, the emphasis is on precision. Under the analogy, restriction in the range of ability is the corollary of tailored test precision. Thus, the most informative, available item is the one that will most restrict the range of ability associated with its estimate, or, synonomously, render the most precise estimate of ability. In this context an item is referred to as available when it has not yet been used in the tailoring of a test for the particular individual. Once the designated level of precision has been achieved, the tailored test is terminated. Equality of precision is then enforced through the appropriate termination of the tailored tests for all individuals.

In constructing the sequence of items for a tailored test, each item is chosen on the basis of the expectation that the individual's answer to that particular item will restrict the variance of ability for the associated estimate more than will the individual's answer to any other available item in the ability bank. Thus, an expected value for the variance of ability is involved. In determining the most informative, available item, this expected value is evaluated for each of the available items. One seeks the item with the minimum value because it is desired to restrict this variance most severely given a sequence of available items. Alternatively, one might state that it is desired to arrive at the designated level of precision with the sequence that consists of as few items as possible given those available. The cumulative effect across all tailored tests is that this method of choosing items is expected to achieve the designated level of reliability for the "scores" obtained from the tailoring process with as few items as possible given the particular ability bank.

One begins by choosing the first item. It is known that the individual being tested is a member of the population of interest. One's estimate of this individual's ability is, then, $\mu(\Theta^*)$, and the variance of this estimate is, then, $\sigma^2(\Theta^*)$. One knows the item parameters a_g^* and b_g^*, in the two-parameter case, or a_g^*, b_g^*, and c_g^*, in the three-parameter case. A formula for the expected value of the variance of ability is now needed which anticipates the probable outcomes of the encounter between this individual and each item g in the ability bank. For convenience in exposition, this expected value will be generalized with respect to the submodels. The statistical rationale basic to choosing the item is shared by both submodels. In the following section, submodel specificity will be provided in the form of computationally convenient identities of this generalized expected value. The purpose in this subsection is to provide the common statistical rationale in explicit form.

Let U_g^* be designated as a generalized binary random variable denoting either U_g or U_g' depending on the submodel context. The generalized expected value for the choice of the first item, then, is

$$\mathscr{E}_{U_g^*}\, \sigma^2(\Theta^* \mid U_g^*) = \Pr(u_g^* = 1)\, \sigma^2(\Theta^* \mid u_g^* = 1) + \Pr(u_g^* = 0)\, \sigma^2(\Theta^* \mid u_g^* = 0) \qquad (5.2.1)$$

where $\Pr(u_g^* = 1)$ and $\Pr(u_g^* = 0)$ are merely the respective probabilities of a correct and incorrect answer to item g given what is currently known about this individual, and $\sigma^2(\Theta^* \mid u_g^* = 1)$ or $\sigma^2(\Theta^* \mid u_g^* = 0)$ is the variance of ability given either of the two probable outcomes, a correct answer or an incorrect answer to item g. This expected value is merely a weighted sum, the sum of weighted variances. Prior to summation, the weights, the probabilities of the particular outcomes, are each multiplied by the variance associated with the particular outcome.

In choosing the first item, a submodel-specific and computationally convenient identity of (5.2.1) (to be provided in Section 5.3) is actually employed. Each item in the ability bank is evaluated. The item g that yields the minimum value for the expected variance is chosen as the first item in the tailoring sequence. This item g is designated as item $g^{(1)}$ to indicate that this item g was chosen as the first item in the tailoring sequence. Such superscription cross-references the item as indexed within the ability bank as the first item in the tailored test. Items within the tailored test are sequentially ordered by the subscript n. For this particular item g, $g^{(1)}$, its subscript within the tailored test, then, is 1. According to the statistical rationale of the choice of this item, the item is by expectation the most informative. The information yielded by the binary score for this particular individual on this particular item is expected to provide the most severe restriction in the variance of ability or, synonomously, to result in the most precise estimate of this individual's ability given the items in the ability bank.

To provide submodel specificity for (5.2.1), one should note that U_g^* has the following realizations:

For the two-parameter submodel,

$u_g = 1$ (answer correct)

and

$u_g = 0$ (answer incorrect);

For the three-parameter submodel,

$u_g' = 1$ (answer correct)

and

$u_g' = 0$ (answer incorrect).

Thus, for the two-parameter normal ogive submodel, substitutions from (2.2.1) and (2.2.3) into (5.2.1) yield

$$\mathscr{E}_{U_g}\, \sigma^2(\Theta^* \mid U_g) = P_g\, \sigma^2(\Theta^* \mid u_g = 1) + Q_g\, \sigma^2(\Theta^* \mid u_g = 0); \qquad (5.2.2)$$

and substitutions from (3.2.1) and (3.2.2) into (5.2.1) provide

$$\mathscr{E}_{U_g'}\, \sigma^2(\Theta^* \mid U_g) = P_g'\, \sigma^2(\Theta^* \mid u_g' = 1) + Q_g'\, \sigma^2(\Theta^* \mid u_g' = 0) \qquad (5.2.3)$$

for the three-parameter normal ogive submodel. Subscripts on the various terms for individual i are omitted because such subscription is implied by context. In (5.2.2) and (5.2.3), submodel specificity is present, but computational convenience awaits the further developments in Section 5.3.

The generalized expected value of the variance of ability has a recursive form. This recursive form is based on the same statistical rationale of item choice discussed in connection with (5.2.1). Identities of this form are used in choosing the nth item in the tailored sequence where $n = 1, 2 \ldots q_i$. This generalized recursive form is given by

$$\begin{aligned}\mathscr{E}_{U_g^*}\, \sigma^2(\Theta^* \mid v_{n-1}^*, U_g^*) = {} & \Pr(u_g^* = 1 \mid v_{n-1}^*)\, \sigma^2(\Theta^* \mid v_{n-1}^*, u_g^* = 1) \\ & + \Pr(u_g^* = 0 \mid v_{n-1}^*)\, \sigma^2(\Theta^* \mid v_{n-1}^*, u_g^* = 0).\end{aligned} \qquad (5.2.4)$$

The form (5.2.4) resembles (5.2.1) except that all terms are modified by the conditional and generalized outcome vector v_{n-1}^*. The conditional and generalized outcome vector, v_{n-1}^*, contains the binary scores for the answers to the $(n-1)$ previously chosen items, along with their original subscripts within the ability bank. In this context, the entry v_{n-1}^*, for n equal to *one* represents the condition at the beginning of testing. Thus, v_0^* is a null entry, implied but often unwritten. The recursive form of the generalized expected value takes into account all the information previously obtained through the tailored testing of this particular individual where the chosen items accessed optimum information in terms of the reduction of the squared error or the minimization of the statistical expectation of the error variance in ability estimation.

Identities of (5.2.4) that are both submodel specific and computationally convenient will be provided in the next section. The purposes in introducing (5.2.4) at this juncture is to complete the discussion of the statistical rationale basic to the choice of items in the tailored testing context.

5.3 Developments from Selection Theory

Selection theory was introduced in a landmark publication by Lawley (1943). This theory later guided important developments by Gulliksen (1950) in the treatment of the effects of variable selection on reliability and validity. Here, this theory provides the basis for developments leading to tailoring algorithms for both the two- and three-parameter normal ogive submodels.

For a very readable and comprehensive discussion of selection theory, the interested reader should consult Lord and Novick (1968). Their discussion provides an excellent background for an understanding of the developments leading to the tailored testing algorithms.

In selection theory, several important points emerge. Two of these points are particularly important from the standpoint of tailored testing. First, any specifiable method of explicit selection is permissible. This point has implications with respect to terminology. In earlier discussion, explicit selection and explicit rejection were considered as analogues of the realizations of the U_g and the U'_g. However, since any specifiable method of explicit rejection might then also be used as a method of explicit selection, the distinction between explicit selection and explicit rejection becomes unnecessary in derivational contexts. In general discussion, this distinction will still be used to promote understanding. When the general meaning is intended mainly in derivational contexts, the terminology *explicit selection* will appear in italicized form. Also to appear in italicized form will be the terminology *incidental selection*, which similarly subsumes in meaning incidental selection and incidental rejection because the change in perspective is permissible given the theory. That any method of explicit selection is permissible is advantageous. This permissibility allows the use of the notation U_g^* for the generalized binary random variable that provides the analogues of *explicit selection*. Submodel specificity and binary score contingency can be introduced after key derivations have been obtained. Second, successive *explicit selections* can be summarized in a single matrix operation. This particularly important point allows developments leading to termination rules for the tailored testing process. These developments are presented in Chapter 7 and involve derivations which indicate the effect of termination on the reliability of the results of the tailored testing process.

The requirements of selection theory may be succinctly summarized in two basic assumptions. The first assumption is that the regression of variables effected by *incidental selection* on variables of *explicit selection* is linear; and the second assumption is that the partial variance-covariance matrix for the variables effected by *incidental selection*, given the variables of *explicit selection*, is homoscedastic.

Generalized expressions for the choice of items, the estimation of ability, and the estimation of the variance of ability will now be derived from selection theory for the first two items in a tailored test. These expressions will then be rendered submodel specific as well as contingent upon the binary scores. In the following chapter these expressions will be summarized. Recursive forms will then be induced that are appropriate for any item with subscript n.

The Three-Variable Selection Problem

The three-variable selection problem provides the setting in which answers are obtained to the following questions:

1. Which item should be given first?
2. What is the estimate of ability given the binary score on this first item?
3. What is the variance of ability given the binary score on this first item?
4. Can this process of choosing and of estimating ability and its variance be extended to a second item?

In answering these questions generalized expressions are derived from the assumptions underlying the three-variable selection problem. These expressions are then rendered in specific form by submodel and binary score. In the three-variable selection problem, *explicit selection* on a predictor variable Z_2 is to occur after *explicit selection* on a predictor variable Z_1. *Incidental selection* on the predictor variable Z_2 and the criterion variable Θ^* was imposed through *explicit selection* on the predictor variable Z_1. There will be further *incidental selection* with respect to the criterion variable Θ^* through subsequent *explicit selection* on the predictor variable Z_2.

Assumptions. Three subsidiary assumptions are involved: (a) the regressions of the predictor variable Z_2 and the criterion variable Θ^* on the predictor variable Z_1 remain unchanged by *explicit selection* on predictor variable Z_1; (b) the partial variances of the predictor variable Z_2 and the criterion variable Θ^* remain unchanged by *explicit selection* on the partialled predictor variable Z_1; and (c) the partial correlation between the predictor variable Z_2 and the criterion variable Θ^* is unaltered by *explicit selection* on the partialled predictor variable Z_1. These subsidiary assumptions are consequences of the two basic assumptions presented earlier.

Assumption (a) is algebraically expressed in

$$\rho(Z_1,Z_2 \mid v_1^*) \frac{\sigma(Z_2 \mid v_1^*)}{\sigma(Z_1 \mid v_1^*)} = \rho(Z_1,Z_2) \frac{\sigma(Z_2)}{\sigma(Z_1)} \tag{5.3.1}$$

and

$$\rho(Z_1,\Theta \mid v_1^*) \frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_1 \mid v_1^*)} = \rho(Z_1,\Theta) \frac{\sigma(\Theta^*)}{\sigma(Z_1)}, \tag{5.3.2}$$

where the regression coefficients on either side of the equal sign in both equations are presented as a product of a correlation coefficient and a ratio of standard deviations. The conditional expression v_1^* is the generalized outcome vector, now with one entry indicating whether Item 1 was in free-response or multiple-choice form, what the binary score was, and which item g, $g^{(1)}$, from the ability bank was selected as Item 1. This binary score and item identification allow complete definition of the specific method of *explicit selection* that occurred on Z_1. Accordingly, $\rho(Z_1,Z_2 \mid v_1^*)$ and $\rho(Z_1,\Theta \mid v_1^*)$ are the respective correlations between Z_1 and Z_2 and between Z_1 and Θ^* after *explicit selection* on Z_1 as denoted by v_1^*. Similarly, the $\sigma(Z_1 \mid v_1^*)$, $\sigma(Z_2 \mid v_1^*)$, and $\sigma(\Theta^* \mid v_1^*)$ are the standard deviations of the indicated variables after *explicit selection* on Z_1. The other terms are as defined earlier.

Assumption (b) is algebraically expressed in

$$\sigma^2(Z_2 \mid v_1^*)\,[1 - \rho^2(Z_1,Z_2 \mid v_1^*)] = \sigma^2(Z_2)\,[1 - \rho^2(Z_1,Z_2)] \tag{5.3.3}$$

and

$$\sigma^2(\Theta^* \mid v_1^*)\,[1 - \rho^2(Z_1,\Theta \mid v_1^*)] = \sigma^2(\Theta^*)\,[1 - \rho^2(Z_1,\Theta)]. \tag{5.3.4}$$

The expressions on either side of the equal signs in both (5.3.3) and (5.3.4) are partial variances. In (5.3.3), the partial variances are those for Z_2 after and before *explicit selection* on Z_1, where Z_1 is the partialled variable. In (5.3.4) the partial variances are those for Θ^* after and before *explicit selection* on Z_1, where Z_1 is the partialled variable.

Assumption (c) is algebraically expressed in

$$\frac{\rho(Z_2,\Theta \mid v_1^*) - \rho(Z_1,Z_2 \mid v_1^*)\,\rho(Z_1,\Theta \mid v_1^*)}{\sqrt{1 - \rho^2(Z_1,Z_2 \mid v_1^*)}\,\sqrt{1 - \rho^2(Z_1,\Theta \mid v_1^*)}} = \frac{\rho(Z_2,\Theta) - \rho(Z_1,Z_2)\,\rho(Z_1,\Theta)}{\sqrt{1 - \rho^2(Z_1,Z_2)}\,\sqrt{1 - \rho^2(Z_1,\Theta)}} \tag{5.3.5}$$

where the expressions on either side of the equal sign are the partial correlations between Z_2 and Θ^* where Z_1 is the partialled variable. Reading from left to right in (5.3.5), one has that the partial correlation after equals the partial correlation before *explicit selection* on Z_1, where the specific method of *explicit selection* is indicated by v_1^*.

The relationships entailed in (5.3.2) and (5.3.4) can be examined in Figure 2.3.1 and Figure 3.3.1 for the specific case where Θ^* equals Θ, or where Θ^* is in standard score form. It was noted earlier that the means and variances of the conditional distributions of Θ given ζ_g were undisturbed given either outcome or realization of U_g or U_g'. The particulars of the outcome are now recorded in v_1^*. Because the means that define the regressions of Θ on ζ_g are undisturbed and the undisturbed variances of Θ given ζ_g are identically the partial variances where Z_g is the partialled variable, these assumptions have been anticipated by prior assumptions. The prior assumptions are those of a normal distribution of ability and the normal ogive regression of U_g or U_g' on ability, which in turn define the joint distributions of the Z_g and Θ as bivariate normal. As seen in Figure 2.3.1 and Figure 3.3.1, the relationships given in (5.3.2) and (5.3.4) are merely the consequences of *explicit selection* given bivariate normality.

In regard to Figure 2.3.1 and Figure 3.3.1, imagine a rescaling of Θ that results in a given Θ^*. This rescaling would merely change the slope of the regressions and alter the magnitude of the partial variances where the Z_g are the partialled variables. Subsequently, *explicit selection* on the Z_g would still leave the regressions and the partial variances undisturbed as indicated in (5.3.2) and (5.3.4)

Which Item First? At this point, each item g in the ability bank is potentially the first item in the tailored test. Which Z_g becomes Z_1 is then the question. Thus the possible entries in v_1^* must be probabalistically anticipated in order to obtain the most informative entry. To do this, one rewrites (5.3.4) as

$$\sigma^2(\Theta^* \mid U_g^*)\,[1 - \rho^2(Z_g,\Theta \mid U_g^*)] = \sigma^2(\Theta^*)\,[1 - \rho^2(Z_g,\Theta)], \tag{5.3.6}$$

which may be rearranged and transposed as

$$\sigma^2(\Theta^* \mid U_g^*) = \sigma^2(\Theta^*) \, [1 - \rho^2(Z_g,\Theta)] + \sigma^2(\Theta^* \mid U_g^*) \, \rho^2(Z_g,\Theta \mid U_g^*). \tag{5.3.7}$$

Also, since any Z_g is potentially Z_1 and v_1^* must be anticipated, one rewrites (5.3.2) as

$$\rho(Z_g,\Theta \mid U_g^*) \frac{\sigma(\Theta^* \mid U_g^*)}{\sigma(Z_g \mid U_g^*)} = \rho(Z_g,\Theta) \frac{\sigma(\Theta^*)}{\sigma(Z_g)}, \tag{5.3.8}$$

where it is to be noted from prior discussion that the standard deviations of the Z_g, the $\sigma(Z_g)$, are all unity. After this substitution for $\sigma(Z_g)$, and the squaring of both sides of (5.3.8), it is known that

$$\rho^2(Z_g,\Theta \mid U_g^*) \frac{\sigma^2(\Theta^* \mid U_g^*)}{\sigma^2(Z_g \mid U_g^*)} = \rho^2(Z_g,\Theta) \, \sigma^2(\Theta^*), \tag{5.3.9}$$

where an explicit solution for $\rho^2(Z_g,\Theta \mid U_g^*)$ yields

$$\rho^2(Z_g,\Theta \mid U_g^*) = \rho^2(Z_g,\Theta) \frac{\sigma^2(Z_g \mid U_g^*) \, \sigma^2(\Theta^*)}{\sigma^2(\Theta^* \mid U_g^*)} \tag{5.3.10}$$

which may be substituted into (5.3.7). After this substitution, one has, through rearranging terms,

$$\sigma^2(\Theta^* \mid U_g^*) = \sigma^2(\Theta^*) \, \{1 - \rho^2(Z_g,\Theta) \, [1 - \sigma^2(Z_g \mid U_g^*)]\} \tag{5.3.11}$$

as the variance of ability for the realizations of U_g^* which can be rendered both submodel and outcome specific.

For the two-parameter normal ogive submodel, or the case where $U_g^* = U_g$, where a correct answer to item g, $(u_g = 1)$, is the anticipated outcome, a substitution from (2.2.7) into (5.3.11), along with some rearrangement, yields

$$\sigma^2(\Theta^* \mid u_g = 1) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta) \frac{\phi(\gamma_g)}{P_g} \left[\frac{\phi(\gamma_g)}{P_g} - \gamma_g \right] \right\} \tag{5.3.12}$$

as the variance of ability given the correct answer. Under the same submodel and in anticipation of an incorrect answer to item g, $(u_g = 0)$, a substitution from (2.2.8) into (5.3.11), along with some rearrangement, provides

$$\sigma^2(\Theta^* \mid u_g = 0) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta) \frac{\phi(\gamma_g)}{Q_g} \left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g \right] \right\} \tag{5.3.13}$$

as the variance of ability given the incorrect answer.

For the two-parameter normal ogive submodel, where $U_g^* = U_g$, substitutions from (5.3.12) and (5.3.13) into (5.2.2) move one closer to an answer to the query: Which item first? After these substitutions, it is known that

$$\begin{aligned} \mathscr{E}_{U_g} \sigma^2(\Theta^* \mid U_g) = {} & P_g \, \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta) \frac{\phi(\gamma_g)}{P_g} \left[\frac{\phi(\gamma_g)}{P_g} - \gamma_g \right] \right\} \\ & + Q_g \, \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta) \frac{\phi(\gamma_g)}{Q_g} \left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g \right] \right\}, \end{aligned} \tag{5.3.14}$$

which can be simplified. Multiplying through both expressions in braces on the right side of (5.3.14) by their respective probabilities, P_g and Q_g, and by rearranging and combining these expressions, one obtains

$$\begin{aligned} & \mathscr{E}_{U_g} \sigma^2(\Theta^* \mid U_g) = \\ & \sigma^2(\Theta^*) \left\{ P_g - \rho^2(Z_g,\Theta) \frac{[\phi(\gamma_g)]^2}{P_g} + \rho^2(Z_g,\Theta) \, \phi(\gamma_g) \, \gamma_g + Q_g - \rho^2(Z_g,\Theta) \frac{[\phi(\gamma_g)]^2}{Q_g} - \rho^2(Z_g,\Theta) \, \phi(\gamma_g) \, \gamma_g \right\}, \end{aligned} \tag{5.3.15}$$

where P_g and Q_g are complementary terms adding to 1, and the third and sixth expressions within the braces cancel each other. Thus, one may write

$$\mathscr{E}_{U_g} \sigma^2(\Theta^* \mid U_g) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta) \, [\phi(\gamma_g)]^2 \left[\frac{1}{P_g} + \frac{1}{Q_g} \right] \right\}, \tag{5.3.16}$$

where the two fractions can be provided a common denominator. It is then known that

$$\mathscr{E}_{U_g}\,\sigma^2(\Theta^* \mid U_g) = \sigma^2(\Theta^*)\left\{1 - \rho^2(Z_g,\Theta)\,[\phi(\gamma_g)]^2\left[\frac{Q_g}{P_gQ_g} + \frac{P_g}{P_gQ_g}\right]\right\}. \tag{5.3.17}$$

But P_g and Q_g are complementary terms adding to 1, which leads to the computationally convenient solution

$$\mathscr{E}_{U_g}\,\sigma^2(\Theta^* \mid U_g) = \sigma^2(\Theta^*)\left\{1 - \rho^2(Z_g,\Theta)\,\frac{[\phi(\gamma_g)]^2}{P_gQ_g}\right\}. \tag{5.3.18}$$

The item g for which (5.3.18) is a minimum is, then, the most informative first item given the two-parameter normal ogive submodel. Note that the variance of ability, $\sigma^2(\Theta^*)$, is a constant in the evaluation of all items with subscript g. Thus the maximum over g of the second term within the braces in (5.3.18),

$$\max_g\ \rho^2(Z_g,\Theta)\,\frac{[\phi(\gamma_g)]^2}{P_gQ_g}, \tag{5.3.19}$$

will yield an identical g to that provided by the full evaluation of (5.3.18) for all g; and, obviously, the minimum over g of the reciprocal of the second term within the braces in (5.3.18),

$$\min_g\ \frac{P_gQ_g}{\rho^2(Z_g,\Theta)\,[\phi(\gamma_g)]^2}, \tag{5.3.20}$$

will yield the same g as would the full evaluation of (5.3.18) for all g. Either (5.3.19) or (5.3.20) may then be used to choose the most informative first item. While the quantities provided by (5.3.18) yield more information on the items, either (5.3.19) or (5.3.20) will provide an identical choice of item. The use of either will introduce a computational savings.

For the three-parameter normal ogive submodel, or the case where $U_g^* = U_g'$, when a correct answer to item g, ($u_g' = 1$), is the anticipated outcome, a substitution from (3.2.5) into (5.3.11), along with some rearrangement, provides

$$\sigma^2(\Theta^* \mid u_g' = 1) = \sigma^2(\Theta^*)\left\{1 - \rho^2(Z_g,\Theta)\,\frac{(1 - c_g)\,\phi(\gamma_g)}{P_g'}\left[\frac{(1 - c_g)\,\phi(\gamma_g)}{P_g'} - \gamma_g\right]\right\} \tag{5.3.21}$$

as the variance of ability given the correct answer. For the same submodel and in anticipation of an incorrect answer to item g, ($u_g' = 0$), a substitution from (3.2.6) into (5.3.11), along with some rearrangement, yields

$$\sigma^2(\Theta^* \mid u_g' = 0) = \sigma^2(\Theta^*)\left\{1 - \rho^2(Z_g,\Theta)\,\frac{\phi(\gamma_g)}{Q_g}\left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g\right]\right\} \tag{5.3.22}$$

as the variance of ability given the incorrect answer.

For the three-parameter normal ogive submodel, where $U_g^* = U_g'$, substitutions from (5.3.21) and (5.3.22) into (5.2.3) move one closer to an answer to the query: Which item first? After these substitutions, it is known that

$$\begin{aligned}\mathscr{E}_{U_g'}\,\sigma^2(\Theta^* \mid U_g') &= P_g'\,\sigma^2(\Theta^*)\left\{1 - \rho^2(Z_g,\Theta)\,\frac{(1 - c_g)\,\phi(\gamma_g)}{P_g'}\left[\frac{(1 - c_g)\,\phi(\gamma_g)}{P_g'} - \gamma_g\right]\right\}\\ &\quad + Q_g'\,\sigma^2(\Theta^*)\left\{1 - \rho^2(Z_g,\Theta)\,\frac{\phi(\gamma_g)}{Q_g}\left[\frac{\phi(\gamma_g)}{Q_g} + \gamma_g\right]\right\},\end{aligned} \tag{5.3.23}$$

which can be simplified. Multiplying through both expressions in braces on the right side of (5.3.23) by their respective probabilities, P_g' and Q_g', and rearranging and combining these expressions, one obtains

$$\begin{aligned}&\mathscr{E}_{U_g'}\,\sigma^2(\Theta^* \mid U_g')\\ &\quad = \sigma^2(\Theta^*)\left\{P_g' - \rho^2(Z_g,\Theta)\,(1 - c_g)^2\,\frac{[\phi(\gamma_g)]^2}{P_g'} + \rho^2(Z_g,\Theta)\,(1 - c_g)\,\phi(\gamma_g)\,\gamma_g\right.\\ &\qquad \left. + Q_g' - Q_g'\,\rho^2(Z_g,\Theta)\,\frac{[\phi(\gamma_g)]^2}{Q_g^2} - Q_g'\,\rho^2(Z_g,\Theta)\,\frac{\phi(\gamma_g)}{Q_g}\,\gamma_g\right\}.\end{aligned} \tag{5.3.24}$$

But P'_g and Q'_g are complementary terms that add to 1, and substitutions from (3.2.2) into the fifth and sixth terms within the braces in (5.3.24) allow further changes toward simplification. After this addition and these substitutions, one may write

$$\mathscr{E}_{U'_g}\, \sigma^2(\Theta^* \mid U'_g) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta)\,(1 - c_g)^2 \frac{[\phi(\gamma_g)]^2}{P'_g} + \rho^2(Z_g,\Theta)\,(1 - c_g)\,\phi(\gamma_g)\,\gamma_g - \rho^2(Z_g,\Theta)\,(1 - c_g) \frac{[\phi(\gamma_g)]^2}{Q_g} - \rho^2(Z_g,\Theta)\,(1 - c_g)\,\phi(\gamma_g)\,\gamma_g \right\}, \quad (5.3.25)$$

where the third and fifth terms within the braces in (5.3.25) now cancel each other. The fourth term within the braces in (5.3.25) may now be multiplied in both the numerator and denominator by $(1 - c_g)$. Thereafter, a substitution from (3.2.2) into the fourth term yields

$$\mathscr{E}_{U'_g}\, \sigma^2(\Theta^* \mid U'_g) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta)\,(1 - c_g)^2 \frac{[\phi(\gamma_g)]^2}{P'_g} - \rho^2(Z_g,\Theta)\,(1 - c_g)^2 \frac{[\phi(\gamma_g)]^2}{Q'_g} \right\}, \quad (5.3.26)$$

where the second and third terms can now be combined, providing

$$\mathscr{E}_{U'_g}\, \sigma^2(\Theta^* \mid U'_g) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta)\,(1 - c_g)^2\,[\phi(\gamma_g)]^2 \left[\frac{1}{P'_g} + \frac{1}{Q'_g} \right] \right\}. \quad (5.3.27)$$

The two fractions in (5.3.27) can now be provided a common denominator. One then has

$$\mathscr{E}_{U'_g}\, \sigma^2(\Theta^* \mid U'_g) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta)\,(1 - c_g)^2\,[\phi(\gamma_g)]^2 \left[\frac{Q'_g}{P'_g Q'_g} + \frac{P'_g}{P'_g Q'_g} \right] \right\}. \quad (5.3.28)$$

But P'_g and Q'_g are complementary terms that add to 1. Thus it is known that

$$\mathscr{E}_{U'_g}\, \sigma^2(\Theta^* \mid U'_g) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta)\,(1 - c_g)^2 \frac{[\phi(\gamma_g)]^2}{P'_g Q'_g} \right\}, \quad (5.3.29)$$

which may be written as

$$\mathscr{E}_{U'_g}\, \sigma^2(\Theta^* \mid U'_g) = \sigma^2(\Theta^*) \left\{ 1 - \rho^2(Z_g,\Theta)\,(1 - c_g) \frac{[\phi(\gamma_g)]^2}{P'_g Q_g} \right\} \quad (5.3.30)$$

because of (3.2.2). In (5.3.30), one now has a computationally convenient solution for the evaluation of each item g. That item g for which (5.3.30) is a minimum is, then, the most informative first item given the three-parameter normal ogive submodel. Again note that the variance of ability, $\sigma^2(\Theta^*)$, is a constant in the evaluation of all items with subscript g. Thus the maximum over g of the second term within the braces in (5.3.30)

$$\max_g \; \rho^2(Z_g,\Theta)\,(1 - c_g) \frac{[\phi(\gamma_g)]^2}{P'_g Q_g} \quad (5.3.31)$$

will yield an identical g to that provided by the full evaluation of (5.3.30) for all g; and, obviously, the minimum over g of the reciprocal of the second term within the braces in (5.3.30),

$$\min_g \; \frac{P'_g Q_g}{\rho^2(Z_g,\Theta)\,(1 - c_g)\,[\phi(\gamma_g)]^2}, \quad (5.3.32)$$

will yield the same g as would the full evaluation of (5.3.30) for all g. Either (5.3.31) or (5.3.32) may then be used to choose the first item. While the quantities provided by (5.3.30) are more informative with respect to each item, either (5.3.31) or (5.3.32) may be used to choose the identical item. The use of either will provide a computational savings.

At this juncture, it may be assumed that the first item has been chosen. In the case of the two-parameter normal ogive submodel, one may have used either (5.3.18), (5.3.19), or (5.3.20) for this purpose. In the case of the three-parameter normal ogive submodel, one may have used either (5.3.30), (5.3.31), or (5.3.32) for this purpose. This item is subscripted by n within the tailored test. Because this is the first item in the tailored test, its subscript takes on the value 1.

What Is the Estimate of Ability Given the Binary Score on this First Item? From (2.2.23), (2.2.29), (3.2.15), and (3.2.16), one knows that the generalized expression for the estimator of ability is

$$\mu(\Theta^* \mid U_1^*) = \mu(\Theta^*) + \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\mu(Z_1 \mid U_1^*), \tag{5.3.33}$$

where $\mu(Z_1 \mid U_1^*)$ is weighted by the regression coefficient given on the right side of the equal sign in (5.3.2). Remember that the numerical value of $\sigma(Z_1)$ in (5.3.2) is known to be 1.

One can now derive the estimator of ability for the two-parameter normal ogive submodel, or the case where $U_1^* = U_1$, when a correct answer to Item 1, $(u_1 = 1)$, is the observed outcome. A substitution from (2.2.23) into (5.3.33), while exchanging the alias n for the chosen $g^{(1)}$ provides

$$\mu(\Theta^* \mid u_1 = 1) = \mu(\Theta^*) + \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\mu(Z_1 \mid u_1 = 1), \tag{5.3.34}$$

which may be expressed as

$$\mu(\Theta^* \mid u_1 = 1) = \mu(\Theta^*) + \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\frac{\phi(\gamma_1)}{P_1} \tag{5.3.35}$$

because of (2.2.23). The estimator of ability can also be derived for the same submodel and an incorrect answer to Item 1, $(u_1 = 0)$. A substitution from (2.2.29) into (5.3.33), while exchanging the alias n for the chosen $g^{(1)}$, provides

$$\mu(\Theta^* \mid u_1 = 0) = \mu(\Theta^*) + \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\mu(Z_1 \mid u_1 = 0) \tag{5.3.36}$$

which may be expressed as

$$\mu(\Theta^* \mid u_1 = 0) = \mu(\Theta^*) - \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\frac{\phi(\gamma_1)}{Q_1} \tag{5.3.37}$$

because of (2.2.29). In (5.3.35) or (5.3.37), one has a convenient expression to estimate ability given either outcome in the encounter between the individual and Item 1 under the two-parameter normal ogive submodel.

One can now derive the estimator of ability for the three-parameter normal ogive submodel, or the case where $U_1^* = U_1'$, when a correct answer to Item 1, $(u_1' = 1)$, is the observed outcome. A substitution from (3.2.15) into (5.3.33), while exchanging the alias n for the chosen $g^{(1)}$, yields

$$\mu(\Theta^* \mid u_1' = 1) = \mu(\Theta^*) + \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\mu(Z_1 \mid u_1' = 1), \tag{5.3.38}$$

which may be expressed as

$$\mu(\Theta^* \mid u_1' = 1) = \mu(\Theta^*) + \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,(1 - c_1)\,\frac{\phi(\gamma_1)}{P_1'} \tag{5.3.39}$$

because of (3.2.15). The estimator of ability can also be obtained for the same submodel given an incorrect answer to Item 1, $(u_1' = 0)$. A substitution from (3.2.16) into (5.3.33), while exchanging the alias n for the chosen $g^{(1)}$, yields

$$\mu(\Theta^* \mid u_1' = 0) = \mu(\Theta^*) + \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\mu(Z_1 \mid u_1' = 0), \tag{5.3.40}$$

which may be expressed as

$$\mu(\Theta^* \mid u_1' = 0) = \mu(\Theta^*) - \rho(Z_1,\Theta)\,\sigma(\Theta^*)\,\frac{\phi(\gamma_1)}{Q_1} \tag{5.3.41}$$

because of (3.2.16). In (5.3.39) or (5.3.41), one has a convenient expression to estimate ability for either outcome in the encounter between the individual and Item 1 given the three-parameter normal ogive submodel.

Notice that $\mu(\Theta^* \mid u_1 = 0)$ and $\mu(\Theta^* \mid u_1' = 0)$, as algebraically defined in (5.3.37) and (5.3.41), respectively, are identical. These identities occur because $\mu(Z_1 \mid u_1 = 0)$ and $\mu(Z_1 \mid u_1' = 0)$, as included in (5.3.36) and (5.3.40) respectively, also have identical algebraic definitions. The mean of Z_g for the subpopulation on the interval from negative infinity to γ_g remains undisturbed by success due to guessing. Success due to guessing is assumed to occur at random over this interval of Z_g.

What Is the Variance of Ability Given the Binary Score on the First Item? In deciding which item to choose as the first item in the tailored test, it was necessary to obtain solutions for the variances of ability given the realization of U_g^*. The generalized expression was derived in (5.3.11) and was given submodel and outcome specificity in (5.3.12), (5.3.13), (5.3.21) and (5.3.22). In this context, it is necessary merely to alias the chosen $g^{(1)}$ by n in each of these equations to obtain the appropriate expressions.

Thus, the generalized expression for the variance of ability given the realizations of U_1^* is obtained from (5.3.11) through the aliasing of subscripts as

$$\sigma^2(\Theta^* | U_1^*) = \sigma^2(\Theta^*) \{1 - \rho^2(Z_1,\Theta) [1 - \sigma^2(Z_1 \mid U_1^*)]\}. \tag{5.3.42}$$

For the two-parameter normal ogive submodel, where $U_1^* = U_1$, and a correct answer to Item 1, $(u_1 = 1)$, is the observed outcome, the aliasing of the subscripts in (5.3.12) provides

$$\sigma^2(\Theta^* \mid u_1 = 1) = \sigma^2(\Theta^*) \left\{1 - \rho^2(Z_1,\Theta) \frac{\phi(\gamma_1)}{P_1} \left[\frac{\phi(\gamma_1)}{P_1} - \gamma_1\right]\right\} \tag{5.3.43}$$

as a convenient expression for the estimation of the variance of ability. For the same submodel and an incorrect answer to Item 1, $(u_1 = 0)$, the aliasing of the subscripts in (5.3.13) yields

$$\sigma^2(\Theta^* \mid u_1 = 0) = \sigma^2(\Theta^*) \left\{1 - \rho^2(Z_1,\Theta) \frac{\phi(\gamma_1)}{Q_1} \left[\frac{\phi(\gamma_1)}{Q_1} + \gamma_1\right]\right\} \tag{5.3.44}$$

as a convenient expression for the estimation of the variance of ability.

For the three-parameter normal ogive submodel, where $U_1^* = U_1'$, and a correct answer to Item 1, $(u_1' = 1)$, is the observed outcome, the aliasing of the subscripts in (5.3.21) provides

$$\sigma^2(\Theta^* \mid u_1' = 1) = \sigma^2(\Theta^*) \left\{1 - \rho^2(Z_1,\Theta) \frac{(1 - c_1)\, \phi(\gamma_1)}{P_1'} \left[\frac{(1 - c_1)\, \phi(\gamma_1)}{P_1'} - \gamma_1\right]\right\} \tag{5.3.45}$$

as a convenient expression for the estimation of the variance of ability. For the same submodel and an incorrect answer to Item 1, $(u_1' = 0)$, the aliasing of the subscripts in (5.3.22) yields

$$\sigma^2(\Theta^* \mid u_1' = 0) = \sigma^2(\Theta^*) \left\{1 - \rho^2(Z_1,\Theta) \frac{\phi(\gamma_1)}{Q_1} \left[\frac{\phi(\gamma_1)}{Q_1} + \gamma_1\right]\right\} \tag{5.3.46}$$

as a convenient expression of the estimation of the variance of ability. Notice that the expressions on the right side of equations (5.3.44) and (5.3.46) are identical. This indicates that $\sigma^2(Z_1 \mid U_1^*)$ of (5.3.42) is identical for the realizations $(u_1 = 0)$ and $(u_1' = 0)$. The reason for this identity is that the variance of Z_g for the interval on Z_g from negative infinity to γ_g remains undisturbed by the effect of guessing in the three-parameter normal ogive submodel because success due to guessing is assumed random over this interval.

Can This Process of Choosing and of Estimating Ability and its Variance be Extended to a Second Item? One begins tailoring a test for an individual with $\mu(\Theta^*)$, $\sigma^2(\Theta^*)$, and, consequently, $\sigma(\Theta^*)$, known. These values represent the initial estimate of the individual's ability, its variance and its standard deviation. In the case of the two-parameter normal ogive submodel a_g^* and b_g^* are known for all g. Given a_g^* and $\sigma(\Theta^*)$, $\rho(Z_g,\Theta)$ can be obtained for all g through the use of (2.2.21); given a_g^*, b_g^*, $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$, γ_g can be obtained for all g through the use of (2.2.22); and given γ_g, P_g and Q_g can be obtained for all g through the use of (2.2.1) and (2.2.3). Remember $\phi(\gamma_g)$ is merely the density in the standard normal distribution evaluated at γ_g as given by (2.2.5). Thus, one can use (5.3.18) or, consequently, (5.3.19) or (5.3.20), to choose the first item, because the required inputs for these equations are known or readily obtainable. What is known or readily obtainable at this point is also sufficient for the estimation of ability and the variance of ability. Given the selected item, $g^{(1)}$, its subscript is aliased by n where n equals *one*. Then Item 1 is presented to the individual. If this individual's answer to Item 1 is correct, (5.3.35) is used to estimate this individual's ability, and (5.3.43) is used to estimate the variance of this individual's ability estimate. If the individual's answer to Item 1 is incorrect, (5.3.37) is used to estimate this individual's ability; and (5.3.44) is used to estimate the variance of this individual's ability estimate.

In the case of the three-parameter normal ogive submodel, one also knows a_g^*, b_g^*, and c_g^* for all g. Given a_g^* and $\sigma(\Theta^*)$, $\rho(Z_g,\Theta)$ can be obtained for all g through the use of (2.2.21); given a_g^*, b_g^*, $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$, γ_g can be obtained for all g through the use of (2.2.22); and given γ_g, P_g' and Q_g can be obtained for all g through the use of (3.2.1) and (2.2.3). The density in the standard normal distribution evaluated at γ_g, $\phi(\gamma_g)$, can be obtained by evaluating (2.2.5) for all g. Thus, one can use (5.3.30) or, consequently, (5.3.31) or (5.3.32) to choose the first item because the required inputs for these equations are known or readily obtainable.

What is known or readily obtainable at this point is also sufficient for the estimation of ability and the variance of ability. Given the selected item $g^{(1)}$, its subscript is aliased by n where n equals *one*. Item 1 is then presented to the individual. If this individual's answer to Item 1 is correct, (5.3.39) is used to estimate this individual's ability; and (5.3.45) is used to estimate the variance of this individual's ability estimate. If this individual's answer to Item 1 is incorrect, (5.3.41) is used to estimate this individual's ability; and (5.3.46) is used to estimate the variance of this individual's ability estimate.

The generalized outcome vector v^*_{n-1} by designation records the $(n-1)$ realizations of U^*_{n-1} for $(n-1) = 0,1, \ldots (q_i - 1)$. At this point v^*_1 contains one of four possible entries: $u_{1[g^{(1)}]} = 1$, $u_{1[g^{(1)}]} = 0$, $u'_{1[g^{(1)}]} = 1$, or $u'_{1[g^{(1)}]} = 0$, where the bracketed notation in the subscript denotes the ability bank subscript of the first chosen item. One now seeks the second item. The outputs of (5.3.35), (5.3.37), (5.3.39) and (5.3.41) may be rendered in generalized form as $\mu(\Theta^* \mid v^*_1)$. At this juncture, $\mu(\Theta^* \mid v^*_1)$ and $\sigma^2(\Theta^* \mid v^*_1)$ are known. Also known are the parameters a^*_g and b^*_g, in the two-parameter case, or a^*_g, b^*_g, and c^*_g, in the three-parameter case, for all of the items in the ability bank.

In the following developments it will be established that enough information is known for the choice of item, ability estimation, and the estimation of the variance of ability for a tailored test of two items. These developments follow from the assumptions, those underlying selection theory and the restricted updating procedure.

Under the restricted updating procedure the distributions of $\Theta^*(v^*_1)$, the distributions of Θ^* resulting from *incidental selection* due to *explicit selection* on Z_1, are assumed normal. This assumption provides a very workable approximation, although it is not readily apparent that this should be the case. For instance, it is known that the outcome dependent distributions, the $\Theta^*(v^*_1)$, can be strictly normal only when $\rho(Z_1,\Theta)$ is zero. As $\rho(Z_1,\Theta)$ approaches the limiting value of unity, the distributions of $\Theta^*(v^*_1)$ become identical in form (but not in scale) to those distributions arising from *explicit selection* on the continuous variable Z_1 as implied by the particular binary outcome v^*_1. For the specific case where Θ^* is distributed as Θ and $\rho(Z_1,\Theta)$ approaches unity, the distributions $\Theta^*(v^*_1)$ and $Z_1(v^*_1)$ become identically distributed.

The distributions of $\Theta^*(v^*_1)$, as can be deduced from the earlier development of the normal ogive submodels and the illustrations of the hypothetical relationships underlying them in Figure 2.1.1 for the two-parameter case and in Figure 3.1.1 for the three-parameter case, are nonnormal. A serious question then is "How closely can $\rho(Z_1,\Theta)$ approach this limiting value of unity and still provide for the workability of this approximation?" Since this assumption will be applied sequentially in the restricted updating procedure, a further serious question becomes "How workable is the approximation provided by the assumption of normal distributions for the $\Theta^*(v^*_n)$ as n, the number of items in the tailored test, increases?"

The results obtained in answer to these serious questions tend to be counter-intuitive. The approximation appears particularly good given well-conditioned ability banks where an item is normally chosen with $b^*_n\ (v^*_{n-1})$ close in value to $\mu(\Theta^* \mid v^*_{n-1})$. For instance, when the correlation $\rho(Z_1,\Theta)$ is in the lower .90's, typically a maximum value for this correlation, the assumption still provides a very workable approximation. Also, the assumption provides a very workable approximation in sequential application. Because of the seriousness of these questions and the nature of their answers, results obtained from later developments will now be introduced which provide illustrations of the efficacy of the assumption underlying the restricted updating procedure.

Data from an earlier empirical investigation (Urry, 1977) of the three-parameter normal ogive submodel are relevant. These data are provided in Table 5.3.1. In this table the specific case is treated where Θ^* is assumed as normally distributed with a mean of zero and variance of unity or where Θ^* is distributed as Θ. The particular scaling of Θ^* has no effect on the form of the distributions of $\Theta^*(v'_1)$. In column (1), the sequential order of items, n, is presented for the tailored test given to individual i. The original correlation between Z_n and $\Theta^*, \rho(Z_n,\Theta)$, is given in column (2). The outcome dependent correlation between Z_n and $\Theta^*(v'_{n-1})$, $\rho(Z_n,\Theta \mid v'_{n-1})$, is tabulated in column (3). This correlation is the one modified by the restriction in range implied by the series of *explicit selections* on the continuous variables Z_1 through Z_{n-1}. Notice that this correlation more noticeably departs from the original correlation $\rho(Z_n,\Theta)$, as given in column (2), when the restriction in range becomes more severe or when n, as given in column (1), increases. In columns (4), (5), and (6) the item parameters a_n, b_n, and c_n are provided. As will later be found, the entries in column (3) are obtained from a_n, as given in column (4), and $\sigma(\Theta^* \mid v'_n)$, as given in column (9), through the use of (6.1.7) for the specific case where Θ^* is distributed as Θ. In column (7) the binary outcome u'_n, either a *one* or a *zero* is provided, indicating whether this individual's response to the multiple-choice item n was correct or incorrect. In columns (8) and (9), the tailored test results are presented for individual i. These results are obtained through the use of the tailored testing algorithm derived in Section 6.2. This algorithm requires the assumption that the distributions of $\Theta^*(v'_n)$ or merely $\Theta(v'_n)$, in this situation, are normal. In column (8) the ability estimate, $\mu(\Theta^* \mid v'_n)$, is

Table 5.3.1
The Similarity of Tailored Test Results for Individual i *Where the Tailoring Algorithm Requires or Does Not Require the Assumption of Normal Distributions for the* $\Theta^*(v_n)$

							Assumption of Normal Distributions for the $\Theta^*(v'_n)$			
							Required by the Algorithm		Not Required by the Algorithm	
Item	Parameters					Binary Outcome	Ability Estimate	Error of Estimate	Ability Estimate	Error of Estimate
n (1)	$\rho(Z_n,\Theta)$ (2)	$\rho(Z_n,\Theta \mid v'_{n-1})$ (3)	a_n (4)	b_n (5)	c_n (6)	u'_n (7)	$\mu(\Theta^* \mid v'_n)$ (8)	$\sigma(\Theta^* \mid v'_n)$ (9)	$\mu(\Theta^* \mid v'_n)$ (10)	$\sigma(\Theta^* \mid v'_n)$ (11)
1	.90	.90	2.09	−.12	.18	1	.47	.86	.47	.86
2	.91	.89	2.25	.59	.17	1	.93	.75	.90	.76
3	.89	.83	1.94	.78	.13	1	1.27	.64	1.23	.64
4	.89	.79	1.99	.71	.15	1	1.44	.57	1.39	.57
5	.81	.62	1.39	1.01	.15	1	1.59	.53	1.53	.55
6	.87	.68	1.75	1.65	.24	1	1.77	.50	1.73	.56
7	.82	.58	1.44	1.37	.23	1	1.88	.47	1.86	.54
8	.84	.59	1.56	1.56	.30	1	1.98	.45	1.98	.52
9	.90	.69	2.11	2.26	.30	0	1.80	.39	1.76	.41
10	.82	.49	1.44	1.41	.24	1	1.87	.38	1.84	.39
11	.82	.48	1.44	1.72	.25	1	1.95	.37	1.92	.38
12	.82	.47	1.45	1.87	.26	0	1.80	.34	1.77	.35
13	.82	.44	1.46	1.32	.25	1	1.85	.33	1.82	.33
14	.81	.42	1.40	1.25	.29	1	1.88	.32	1.86	.32
15	.80	.40	1.35	1.93	.28	1	1.94	.32	1.92	.32

provided. This ability estimate is dependent upon the binary outcomes u'_n as denoted by the outcome vector v'_n. Given the binary outcome on Item 1, this ability estimate is the mean of the distribution of $\Theta^*(v'_1)$. For the outcomes on items subsequent to Item 1, it represents an approximation to the mean of the distribution of $\Theta^*(v'_n)$ because of the updating assumption. In column (9) the error of the ability estimate $\sigma(\Theta^* \mid v'_n)$ is given. This error is also dependent upon the binary outcomes u'_n as denoted by the outcome vector v'_n. Given the binary outcome on Item 1, this error of the ability estimate is the standard deviation of the distribution of $\Theta^*(v'_1)$. For the outcomes on items subsequent to Item 1, it represents an approximation to the standard deviation of $\Theta^*(v'_n)$ because of the updating assumption. In columns (10) and (11) the tailored test results for individual i obtained through the use of another algorithm are displayed. This algorithm does not require the assumption that the distributions of $\Theta^*(v'_n)$, or merely $\Theta(v'_n)$, in this situation, are normal. Under this algorithm, the distributions of $\Theta^*(v'_n)$ are obtained using a procedure to be described in Section 9.3. The means, the $\mu(\Theta^* \mid v'_n)$, and the standard deviations, the $\sigma(\Theta^* \mid v'_n)$, can then be calculated. In this algorithm the means and standard deviations are exactly determined. Thus $\mu(\Theta^* \mid v'_n)$ is a least squares estimator of ability and $\sigma(\Theta^* \mid v'_n)$ is the error of this least squares estimator. The means and standard deviations are obtained through numerical integration. In this instance, the integrals required in the calculation of the means and standard deviations were evaluated through the use of Simpson's Rule. Given the required integrals, the means and standard deviations presented in columns (10) and (11) were then calculated.

Since the tailored test results presented in columns (8) and (9) are obtained through the algorithm where the assumption of normal distributions for the $\Theta^*(v'_n)$ is required, one can assess the workability of this approximation by comparing these results with those presented in columns (10) and (11). These latter results were obtained through the algorithm where the assumption of normal distributions for the $\Theta^*(v'_n)$ is not required.

In Table 5.3.1, specifically in columns (8) and (10) for Item 1, it may be noted that the ability estimates are identical in value, .47, across algorithms, given the binary outcome on this item. Also, given the binary outcome on this item, it may be noted that the error of these ability estimates, as given for Item 1 in columns (9) and (11), are also identical in value, .86, across algorithms. Such is the case because the assumption of normality that is basic to the restricted updating procedure has not yet been invoked. Under the restricted updating procedure, normality is assumed in order to continue the estimation of ability and its error after the results have been obtained for the binary outcome on the first item. For Item 2, one can then compare the entry in column (8) with that in column (10) and the entry in column (9) with that in column (11) in order to judge the workability of this approximation. For the ability estimates, the $\mu(\Theta^* \mid v'_2)$, the compared values are .93 and .90; and the compared values are .75 and .76 for the respective errors of these ability estimates, the $\sigma(\Theta^* \mid v_2')$. This similarity in compared values occurs under the condition where $\rho(Z_1, \Theta)$ as indicated in column (2) was .90 prior to the *explicit selection* on the continuous variable Z_1 as implied by the outcome vector v_1'. Thus, it is known that the distribution of $\Theta^*(v_1')$ is, strictly speaking, nonnormal; but the similarity of results indicates that the assumption of normality provides a very workable approximation even when $\rho(Z_1, \Theta)$ closely approaches the limiting value of unity. Subsequently, the correlations of concern are the $\rho(Z_n, \Theta \mid v_{n-1}')$, as given in column (3). These correlations do not approach this limiting value as closely as those given in column (2) because of restriction in range. One then expects less severe departure from normality given the binary outcomes on subsequent items as indicated by the decline in the value of the correlations in column (3) in relation to those in column (2).

Intra-item comparisons can likewise be made for the remaining items in Table 5.3.1. These further comparisons confirm the workable nature of the approximation provided by the assumption underlying the restricted updating procedure. The tailored test results reported for the binary outcome on Item 15 are of particular interest. At this point, the updating assumption has been invoked $(n-1)$ or fourteen times. One can judge the workability of the approximation in sequential application then by comparing the $\mu(\Theta^* \mid v_{15}')$ and the $\sigma(\Theta^* \mid v_{15}')$ across algorithms. For Item 15, one will want to compare: the entry in column (8), or 1.94, with that in column (10), or 1.92; and the entry in column (9), or .32, with that in column (11), or .32; where the only discrepancy is a difference of .02 between $\mu(\Theta^* \mid v_{15}')$ as approximated by the algorithm to be derived in Section 6.2 and $\mu(\Theta^* \mid v_{15}')$ as obtained from the algorithm that does not require the assumption of normality for the $\Theta^*(v_{n-1}')$.

The workability of this approximation can also be assessed in the normative context. The comparisons made in Table 5.3.1 were within the individual and across algorithms. In the normative context, simulation procedures to be described in Chapter 11 may be used to assess the workability of this approximation. In brief, one determines a particular terminal error using the procedures to be described in Chapter 7. Tailored testing is then simulated for a large number of individuals until this particular terminal error is reached. Subsequently, one can correlate the obtained ability estimates with values of true ability which are known given the simulation procedure. Since the terminal error, given the updating assumption, allows a forecast of the correlation between ability estimates and the values of true ability, a comparison of the obtained correlation and the forecasted correlation provides an assessment of the workability of this approximation. If the obtained correlation is sufficiently close in value to the correlation forecasted from the theory, it can be concluded that the assumption provides a workable approximation. This conclusion follows because the forecasting of the correlation requires an unspecifiable but very large number of invocations of the updating assumption. In normative assessments of the efficacy of the normal assumption basic to the restricted updating procedure, it is also found that this assumption provides a very workable approximation. These findings necessarily follow when the individual assessments of this assumption indicate its efficacy.

Since the $\Theta^*(v_1^*)$ can be considered normal for practical purposes, the $Z_2(v_1^*)$, the distribution of Z_2 resulting from *incidental selection* due to *explicit selection* on Z_1 can also be considered normal as a workable approximation. The reason for the workability of this approximation is that $Z_2(v_1^*)$ is a linear combination of ability and a homoscedastic random error variable, which is normally distributed. Remember, the random error variables are assumed under the basic model to be independent from item to item. Thus explicit selection on Z_1 has no effect on the homoscedastic random error in Z_2. As a result, the joint distribution of $Z_2(v_1^*)$ and $\Theta^*(v_1^*)$ is bivariate normal to a workable approximation.

Because bivariate normality is a workable approximation, one should seek convenient expressions for $\rho(Z_2, \Theta \mid v_1^*)$, the correlation between $Z_2(v_1^*)$ and $\Theta^*(v_1^*)$, and $\gamma_2(v_1^*)$, the point of dichotomization on $\tilde{Z}_2(v_1^*)$, or, synonomously, $Z_2(v_1^*)$ after its standardization to a mean of zero and variance of unity. These expressions will allow one to continue with the repeated application of the selection or rejection analogy in the tailored testing context. One will then be considering *explicit selection* on $\tilde{Z}_2(v_1^*)$, standardized $Z_2(v_1^*)$, where the "cut score" is $\gamma_2(v_1^*)$.

Since the derivations leading to convenient expressions for $\rho(Z_2, \Theta \mid v_1^*)$ and $\gamma_2(v_1^*)$ are of considerable length and detail, only the principal results are presented here. These derivations may be found in Sections 5.4. In the course of these derivations, solutions are also found for the parameters $a_2(v_1^*)$, item discriminatory power, and $b_2(v_1^*)$, item difficulty. These parameters are those for Item 2 that are appropriate for $\tilde{\Theta}^*(v_1^*)$, or the continuous variable of ability given the generalized outcome v_1^* in standardized form. These parameters provide the property of invariance.

The presented expressions apply to any potential second item in the tailored test. At this juncture, there are $(p-1)$ items remaining in the ability bank as possible choices for a second item.

In deriving a convenient expression for $\rho(Z_2, \Theta \mid v_1^*)$, it is necessary to obtain $a_2(v_1^*)$, the item parameter of discriminatory power that is appropriate for $\tilde{\Theta}^*(v_1^*)$, the standardized form of the continuous variable of ability subsequent to the generalized outcome v_1^*. The solution for this parameter is provided by

$$a_2(v_1^*) = a_2^* \, \sigma(\Theta^* \mid v_1^*), \tag{5.3.47}$$

where a_2^* is the parameter of item discriminatory power appropriate for Θ^*, as known; and $\sigma(\Theta^* \mid v_1^*)$ is the standard deviation of Θ^* given the generalized outcome v_1^*, as provided by the square root of either (5.3.43) or (5.3.44) in the case of the two-parameter normal ogive submodel, or of either (5.3.45) or (5.3.46) in the case of the three-parameter normal ogive submodel.

A convenient expression for the correlation between Z_2 and Θ^* given the generalized outcome v_1^*, $\rho(Z_2, \Theta \mid v_1^*)$, is provided by

$$\rho(Z_2, \Theta \mid v_1^*) = \frac{a_2(v_1^*)}{\{1 + [a_2(v_1^*)]^2\}^{.5}} = \frac{a_2^* \sigma(\Theta^* \mid v_1^*)}{\{1 + [a_2^* \sigma(\Theta^* \mid v_1^*)]^2\}^{.5}}, \tag{5.3.48}$$

where $a_2(v_1^*)$ is the parameter of item discriminatory power appropriate for $\tilde{\Theta}^*(v_1^*)$, as provided by (5.3.47); a_2^* is the parameter of item discriminatory power appropriate for Θ^*, as known; and $\sigma(\Theta^* \mid v_1^*)$ is the standard deviation of Θ^* given the generalized outcome v_1^*, as provided by the square root of either (5.3.43) or (5.3.44) in the case of the two-parameter normal ogive submodel, or of either (5.3.45) or (5.3.46) in the case of the three-parameter normal ogive submodel.

In deriving a convenient expression for $\gamma_2(v_1^*)$, it is necessary to obtain a solution for $b_2(v_1^*)$, the parameter of item difficulty appropriate for $\tilde{\Theta}^*(v_1^*)$, the continuous variable of ability given the generalized outcome v_1^* in standardized form. The solution for this parameter is provided by

$$b_2(v_1^*) = \frac{b_2^* - \mu(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_1^*)}, \tag{5.3.49}$$

where b_g^* is the parameter of item difficulty appropriate for Θ^*, as known; $\mu(\Theta^* \mid v_1^*)$ is the estimator of ability as provided by either (5.3.35) or (5.3.37) in the case of the two-parameter normal ogive submodel or by either (5.3.39) or (5.3.41) in the case of the three-parameter normal ogive submodel; and $\sigma(\Theta^* \mid v_1^*)$ is the standard deviation of Θ^* given the generalized outcome v_1^*, as provided by the square root of either (5.3.43) or (5.3.44) in the case of the two-parameter normal ogive submodel, or of either (5.3.45) or (5.3.46) in the case of the three-parameter normal ogive submodel.

A convenient expression for the point of dichotomization on $\tilde{Z}_2(v_1^*)$, the standardized form of the continuous variable Z_2 given the generalized outcome v_1^*, is provided by

$$\gamma_2(v_1^*) = \frac{b_2^* - \mu(\Theta^* \mid v_1^*)}{[(a_2^*)^{-2} + \sigma^2(\Theta^* \mid v_1^*)]^{.5}}, \tag{5.3.50}$$

where a_2^* and b_2^* are the parameters of item discriminatory power and item difficulty, respectively, that are appropriate for Θ^*, as known; $\mu(\Theta^* \mid v_1^*)$ is the estimator of ability, as provided by either (5.3.35) or (5.3.37) in the case of the two-parameter normal ogive submodel or by either (5.3.39) or (5.3.41) in the case of the three-parameter normal ogive submodel; and $\sigma^2(\Theta^* \mid v_1^*)$ is the variance of Θ^* given the generalized outcome v_1^*, as provided by either (5.3.43) or (5.3.44) in the case of the two-parameter normal ogive submodel or by either (5.3.45) or (5.3.46) in the case of the three-parameter normal ogive submodel.

As indicated by the asterisk on the generalized outcome v_1^*, the item parameters defined in (5.3.47) and (5.3.49) are those for both the two- and three-parameter normal ogive submodels. These respective parameters are defined in an identical manner under both submodels. In the three-parameter normal ogive submodel, the third item parameter appropriate for $\tilde{\Theta}^*(v_1^*)$, the standardized form of the variable of ability given the outcome v_1', is provided by

$$c_2(v_1') = c_2^* = c_2, \tag{5.3.51}$$

which indicates that the lower asymptote of the characteristic curve for Item 2 remains undisturbed by a linear transformation of the continuous variable of ability.

For both submodels, the given item parameters provide the property of invariance. The probability of a correct answer to Item 2 remains invariant for corresponding values of Θ, Θ^*, and $\bar{\Theta}^*(v_1^*)$. Under these changes in the scale of the continuous variable of ability, the characteristic curve for Item 2 remains invariant.

At this point, it is evident that the process can be extended. This extension of the process consists of choosing the second item and of estimating ability and its variance given the outcome on this chosen item. For the extension of the process, one now turns to the setting of the four-variable selection problem.

The Four-Variable Selection Problem

The four-variable selection problem provides the setting in which the answers are obtained to the following questions:

1. Which item should be given second?
2. What is the estimate of ability given the binary score on this second item?
3. What is the variance of ability given the binary score on this second item?
4. Can this process of choosing and of estimating ability and its variance be extended to a third item?

In answering these questions, generalized expressions are derived from the assumptions underlying the four-variable selection problem. These expressions are then rendered in specific form by submodel and binary score. In the four-variable selection problem, *explicit selection* on a predictor variable Z_3 is to occur after *explicit selections* on the predictor variables Z_1 and Z_2. *Incidental selections* on the predictor variable Z_3 and the criterion variable Θ^* were imposed through *explicit selections* on the predictor variables Z_1 and Z_2. There will be further *incidental selection* with respect to the criterion variable Θ^* through subsequent *explicit selection* on the predictor variable Z_3.

Assumptions. Three subsidiary assumptions are involved: (a) the regressions of the predictor variable Z_3 and the criterion variable Θ^* on the predictor variable Z_2 remain unchanged by *explicit selections* on the predictor variables Z_1 and Z_2; (b) the partial variances of the predictor variable Z_3 and the criterion variable Θ^*, where the partialled predictor variable is Z_2, remain unchanged by *explicit selections* on the predictor variables Z_1 and Z_2; and (c) the partial correlation between the predictor variable Z_3 and the criterion variable Θ^* with Z_2 as the partialled predictor variable remains unaltered by *explicit selections* on the predictor variables Z_1 and Z_2. These subsidiary assumptions are consequences of the two basic assumptions presented earlier.

Assumption (a) is algebraically expressed in

$$\rho(Z_2,Z_3 \mid v_2^*)\,\frac{\sigma(Z_3 \mid v_2^*)}{\sigma(Z_2 \mid v_2^*)} = \rho(Z_2,Z_3 \mid v_1^*)\,\frac{\sigma(Z_3 \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)} = \rho(Z_2,Z_3)\,\frac{\sigma(Z_3)}{\sigma(Z_2)} \tag{5.3.52}$$

and

$$\rho(Z_2,\Theta \mid v_2^*)\,\frac{\sigma(\Theta^* \mid v_2^*)}{\sigma(Z_2 \mid v_2^*)} = \rho(Z_2,\Theta \mid v_1^*)\,\frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)} = \rho(Z_2,\Theta)\,\frac{\sigma(\Theta^*)}{\sigma(Z_2)}\,, \tag{5.3.53}$$

where the three equalities in both equations are regression coefficients presented as the product of a correlation coefficient and a ratio of standard deviations. The conditional expression v_2^* is the generalized outcome vector, now with two entries: v_1^* as defined earlier and an entry indicating whether Item 2 was in free-response or multiple-choice form, what the binary score was, and which item g, $g^{(2)}$, from the ability bank was chosen as Item 2. This binary score and item identification allow complete definition of the specific method of *explicit selection* that occurred on $\bar{Z}_2(v_1^*)$. Accordingly, $\rho(Z_2,Z_3 \mid v_2^*)$ and $\rho(Z_2,\Theta \mid v_2^*)$ are the respective correlations between Z_2 and Z_3 and between Z_2 and Θ^* after *explicit selections* on Z_1 and $\bar{Z}_2(v_1^*)$. Correspondingly, $\sigma(Z_2 \mid v_2^*)$, $\sigma(Z_3 \mid v_2^*)$, and $\sigma(\Theta^* \mid v_2^*)$ are the standard deviations of the indicated variables after *explicit selections* on Z_1, and $\bar{Z}_2(v_1^*)$. The other terms are as defined earlier.

Assumption (b) is algebraically expressed in

$$\sigma^2(Z_3 \mid v_2^*)[1-\rho^2(Z_2,Z_3 \mid v_2^*)] = \sigma^2(Z_3 \mid v_1^*)[1-\rho^2(Z_2,Z_3 \mid v_1^*)] = \sigma^2(Z_3)[1-\rho^2(Z_2,Z_3)] \tag{5.3.54}$$

and

$$\sigma^2(\Theta^* \mid v_2^*)[1-\rho^2(Z_2,\Theta \mid v_2^*)] = \sigma^2(\Theta^* \mid v_1^*)[1-\rho^2(Z_2,\Theta \mid v_1^*)] = \sigma^2(\Theta^*)[1-\rho^2(Z_2,\Theta)], \tag{5.3.55}$$

where the three equalities in both equations are partial variances. In (5.3.54) and reading from left to right, the partial variances are those of Z_3: where the partialled variable is Z_2; after *explicit selections* on Z_1 and $\bar{Z}_2(v_1^*)$; after *explicit selection* on Z_1 and before *explicit selection* on $\bar{Z}_2(v_1^*)$; and before any *explicit selection*. In (5.3.55), reading from left to right, the partial variances are those of Θ^* where the partialled variable is Z_2: after *explicit selections* on Z_1 and $\tilde{Z}_2(v_1^*)$; after *explicit selection* on Z_1 and before *explicit selection* on $\tilde{Z}_2(v_1^*)$; and before any *explicit selection*.

Assumption (c) is algebraically expressed in

$$\frac{\rho(Z_3,\Theta \mid v_2^*) - \rho(Z_2,Z_3 \mid v_2^*)\,\rho(Z_2,\Theta \mid v_2^*)}{\sqrt{1-\rho^2(Z_2,Z_3 \mid v_2^*)}\,\sqrt{1-\rho^2(Z_2,\Theta \mid v_2^*)}} = \frac{\rho(Z_3,\Theta \mid v_1^*) - \rho(Z_2,Z_3 \mid v_1^*)\rho(Z_2,\Theta \mid v_1^*)}{\sqrt{1-\rho^2(Z_2,Z_3 \mid v_1^*)}\,\sqrt{1-\rho^2(Z_2,\Theta \mid v_1^*)}}$$

$$= \frac{\rho(Z_3,\Theta) - \rho(Z_2,Z_3)\rho(Z_2,\Theta)}{\sqrt{1-\rho^2(Z_2,Z_3)}\,\sqrt{1-\rho^2(Z_2,\Theta)}} \tag{5.3.56}$$

where the three equalities are the partial correlations between Z_3 and Θ^* where Z_2 is the partialled variable. In (5.3.56) and reading from left to right, this partial correlation is represented: after *explicit selections* on Z_1 and $\tilde{Z}_2(v_1^*)$; after *explicit selection* on Z_1 and before *explicit selection* on $\tilde{Z}_2(v_1^*)$; and before any *explicit selection*.

Some Consequences of the Submodels and the Updating Assumption. At this juncture it is convenient to summarize some intermediate results. These results are derived in Section 5.4 and are required in subsequent derivations. These derivations will be used to develop solutions for the questions posed earlier: Which item should be given second? What is the estimate of ability given the binary score on this second item? and What is the variance of ability given the binary score on the second item?

These intermediate results consist of expressions for: the probabilities of correct and incorrect answers to Item 2 given the generalized outcome v_1^*; the means of $\tilde{Z}_2(v_1^*)$ and $\dot{Z}_2(v_1^*)$ for each of the realizations of the generalized binary random variable U_2^* where $\dot{Z}_2(v_1^*)$ is a mean deviate form of the continuous variable $Z_2(v_1^*)$; and the variances of $\tilde{Z}_2(v_1^*)$ given each of the realizations of the generalized binary random variable U_2^* where $\tilde{Z}_2(v_1^*)$ is the standardized form of the continuous variable Z_2 subsequent to the effects of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the generalized outcome v_1^*.

Given the two-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1^*:

The probability of a correct answer to Item 2, $\Pr(u_2 = 1 \mid v_1)$ or $P_2(v_1)$, is provided by

$$\Pr(u_2 = 1 \mid v_1) = P_2(v_1) = \Pr[\tilde{Z}_2(v_1) \geqslant \gamma_2(v_1)] = \Phi[-\gamma_g(v_1)], \tag{5.3.57}$$

where $\Pr[\tilde{Z}_2(v_1) \geqslant \gamma_2(v_1)]$ is the probability that $\tilde{\zeta}_2(v_1)$ is greater than or equal to $\gamma_2(v_1)$, the point of dichotomization on $\tilde{Z}_2(v_1)$, the standardized form of the continuous variable Z_2 subsequent to the effects of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1. The point of dichotomization, $\gamma_2(v_1)$, is provided by (5.3.50) after setting the generalized outcome v_1^* equal to v_1. In (5.3.57), $\Phi[-\gamma_2(v_1)]$ is the cumulative normal distribution function evaluated for $\tilde{Z}_2(v_1)$ on the interval extending from negative infinity to negative $\gamma_2(v_1)$ or, due to symmetry, the area above $\gamma_2(v_1)$ in the standard normal distribution.

The probability of an incorrect answer to Item 2, $\Pr(u_2 = 0 \mid v_1)$ or $Q_2(v_1)$, is provided by

$$\Pr(u_2 = 0 \mid v_1) = Q_2(v_1) = \Pr[\tilde{Z}_2(v_1) < \gamma_2(v_1)] = \Phi[\gamma_2(v_1)], \tag{5.3.58}$$

where $\Pr[\tilde{Z}_2(v_1) < \gamma_2(v_1)]$ is the probability that $\tilde{\zeta}_2(v_1)$ is less than $\gamma_2(v_1)$, the point of dichotomization on $\tilde{Z}_2(v_1)$, the standardized form of Z_2 subsequent to the effects of *incidental selection* resulting from *explicit selection* on Z_1 as indicated by the outcome v_1. The point of dichotomization is provided by (5.3.50) after setting the generalized outcome v_1^* equal to v_1. In (5.3.58), $\Phi[\gamma_2(v_1)]$ is the cumulative normal distribution function evaluated for that interval of $\tilde{Z}_2(v_1)$ extending from negative infinity to $\gamma_2(v_1)$, or the area below $\gamma_2(v_1)$ in the standard normal distribution.

Given the three-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1':

The probability of a correct answer to Item 2, $\Pr(u_2' = 1 \mid v_1')$ or $P_2'(v_1')$, is provided by

$$\Pr(u_2' = 1 \mid v_1') = P_2'(v_1')$$

$$= \Pr[\tilde{Z}_2(v_1') \geqslant \gamma_2(v_1')] + c_2 \Pr[\tilde{Z}_2(v_1') < \gamma_2(v_1')]$$

$$= c_2 + (1 - c_2)\,\Phi[-\gamma_2(v_1')], \tag{5.3.59}$$

where $\Pr[\tilde{Z}_2(v_1') \geqslant \gamma_2(v_1)]$ is the probability that $\tilde{\zeta}_2(v_1')$ is equal to or greater than $\gamma_2(v_1')$, the point of dichotomization on $\tilde{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effects of *incidental selection* resulting from *explicit selection* on Z_1 as indicated by the outcome v_1'; $\gamma_2(v_1')$, as defined, is provided by (5.3.50) after setting the generalized outcome v_1^* equal to v_1'; c_2 is the coefficient of guessing for Item 2, as known; $\Pr[\tilde{Z}_2(v_1') < \gamma_2(v_1')]$ is the probability that $\tilde{\zeta}_2(v_1')$ is less than $\gamma_2(v_1')$; and $\Phi[-\gamma_2(v_1')]$ is the cumulative normal distribution function evaluated for the interval of $\tilde{Z}_2(v_1')$ extending from negative infinity to negative $\gamma_2(v_1')$ or, due to symmetry, the area above $\gamma_2(v_1')$ in the standard normal distribution.

The probability of an incorrect answer to Item 2, $\Pr(u_2' = 0 \mid v_1')$ or $Q_2'(v_1')$, is provided by

$$\Pr(u_2' = 0 \mid v_1') = Q_2'(v_1')$$

$$= (1 - c_2) \Pr[\tilde{Z}_2(v_1') < \gamma_2(v_1')] = (1 - c_2)\,\Phi[\gamma_2(v_1')], \tag{5.3.60}$$

where c_2 is the coefficient of guessing for Item 2, as known; $\Pr[\tilde{Z}_2(v_1') < \gamma_2(v_1')]$ is the probability that $\tilde{\zeta}_2(v_1')$ is less than $\gamma_2(v_1')$, the point of dichotomization on $\tilde{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on Z_1 as indicated by the outcome v_1'. The point of dichotomization $\gamma_2(v_1')$ is provided by (5.3.50) after setting the generalized outcome v_1^* equal to v_1'. In (5.3.60), $\Phi[\gamma_2(v_1')]$ is the cumulative normal distribution function evaluated for the interval of $\tilde{Z}_2(v_1')$ extending from negative infinity to $\gamma_2(v_1')$, or the area below $\gamma_2(v_1')$ in the standard normal distribution.

Given the two-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1:

The mean of $\tilde{Z}_2(v_1)$, when a correct answer to Item 2 is observed, $\mu(\tilde{Z}_2 \mid v_1, u_2 = 1)$, is provided by

$$\mu(\tilde{Z}_2 \mid v_1, u_2 = 1) = \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)}, \tag{5.3.61}$$

where $\gamma_2(v_1)$ is the point of dichotomization on $\tilde{Z}_2(v_1)$, the standardized form of the continuous variable Z_2 subsequent to the effects of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1; $\gamma_2(v_1)$ is provided by (5.3.50) after setting the generalized outcome v_1^* equal to v_1; $\phi[\gamma_2(v_1)]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1)$, as obtained from (2.2.5) after a substitution of $\gamma_2(v_1)$ for γ_g; and $P_2(v_1)$ is the probability of a correct answer to Item 2 given the outcome v_1, as obtained from (5.3.57).

The mean of $\tilde{Z}_2(v_1)$ when an incorrect answer to Item 2 is observed, $\mu(\tilde{Z}_2 \mid v_1, u_2 = 0)$, is provided by

$$\mu(\tilde{Z}_2 \mid v_1, u_2 = 0) = -\frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)}, \tag{5.3.62}$$

where $\gamma_2(v_1)$ is the point of dichotomization on $\tilde{Z}_2(v_1)$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1, as indicated by the outcome v_1; $\gamma_2(v_1)$ is obtained from (5.3.50) after setting the generalized binary outcome v_1^* equal to v_1; $\phi[\gamma_2(v_1)]$ is the density in the standard normal distribution at $\gamma_2(v_1)$ as provided by (2.2.5) after a substitution of $\gamma_2(v_1)$ for γ_g; and $Q_2(v_1)$ is the probability of an incorrect answer to Item 2 given the binary outcome v_1, as obtained from (5.3.58).

Given the three-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1':

The mean of $\tilde{Z}_2(v_1')$ when a correct answer to Item 2 is observed, $\mu(\tilde{Z}_2 \mid v_1', u_2' = 1)$, is provided by

$$\mu(\tilde{Z}_2 \mid v_1', u_2' = 1) = (1 - c_2) \frac{\phi[\gamma_2(v_1')]}{P_2'(v_1')}, \tag{5.3.63}$$

where c_2 is the coefficient of guessing for Item 2, as known; $\gamma_2(v_1')$ is the point of dichotomization on $\tilde{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on Z_1, as indicated by the binary outcome v_1'; $\gamma_2(v_1')$ is provided by (5.3.50) after setting the generalized binary outcome v_1^* equal to v_1'; $\phi[\gamma_2(v_1')]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1')$ as obtained from (2.2.5) after a substitution of $\gamma_2(v_1')$ for γ_g; and $P_2'(v_1')$ is the probability of a correct answer to Item 2 given the binary outcome v_1' as provided by (5.3.59).

The mean of $\tilde{Z}_2(v_1')$ when an incorrect answer to Item 2 is observed, $\mu(\tilde{Z}_2 \mid v_1', u_1' = 0)$, is provided by

$$\mu(\tilde{Z}_2 \mid v_1', u_2'=0) = - \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')}, \tag{5.3.64}$$

where $\gamma_2(v_1')$ is the point of dichotomization on $\tilde{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1'; $\gamma_2(v_1')$ is obtained from (5.3.50) after setting the binary outcome v_1^* equal to v_1'; $\phi[\gamma_2(v_1')]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1')$, as provided by (2.2.5) after a substitution of $\gamma_2(v_1')$ for γ_g; and $Q_2(v_1')$ is the probability of not recognizing the correct alternative to multiple-choice Item 2 given the binary outcome v_1'. The probability $Q_2(v_1')$ is provided by

$$Q_2(v_1') = \Pr[\tilde{Z}_2(v_1') < \gamma_2(v_1')] = \Phi[\gamma_2(v_1')], \tag{5.3.65}$$

where $\Pr[\tilde{Z}_2(v_1') < \gamma_2(v_1')]$ is the probability that $\tilde{\zeta}_2(v_1')$ is less than $\gamma_2(v_1')$, the point of dichotomization on the continuous variable $\tilde{Z}_2(v_1')$, the standardized form of Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on Z_1 as indicated by the binary outcome v_1'; $\gamma_2(v_1')$ is obtained from (5.3.50) after setting the generalized binary outcome v_1^* equal to v_1'. In (5.3.65), $\Phi[\gamma_2(v_1')]$ is the cumulative normal distribution function evaluated for that interval of $\tilde{Z}_2(v_1')$ extending from negative infinity to $\gamma_2(v_1')$, or the area below $\gamma_2(v_1')$ in the standard normal distribution.

For later developments, it is useful to define the continuous variable $\dot{Z}_2(v_1^*)$ whose realizations, $\dot{\zeta}_2(v_1^*)$,are provided by

$$\dot{\zeta}_2(v_1^*) = \zeta_2 - \mu(Z_2 \mid v_1^*). \tag{5.3.66}$$

This continuous variable is merely the continuous variable Z_2 in mean deviate form about $\mu(Z_2 \mid v_1^*)$, the mean of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on Z_1 as indicated by the generalized binary outcome v_1^*. For this situation, it is known that the mean of the continuous variable $\dot{Z}_2(v_1^*)$ given the realization of the generalized binary random variable U_2^*, $\mu(\dot{Z}_2 \mid v_1^*, U_2^*)$, is provided by

$$\begin{aligned}\mu(\dot{Z}_2 \mid v_1^*, U_2^*) &= \sigma(Z_2 \mid v_1^*)\, \mu\,(\tilde{Z}_2 \mid v_1^*, U_2^*) \\ &= \mu(Z_2 \mid v_1^*, U_2^*) - \mu(Z_2 \mid v_1^*),\end{aligned} \tag{5.3.67}$$

where $\sigma(Z_2 \mid v_1^*)$ is the standard deviation of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the generalized binary outcome v_1^*; $\mu(\tilde{Z}_2 \mid v_1^*, U_2^*)$ is the mean of $\tilde{Z}_2$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the generalized outcome v_1^*, which is then subjected to *explicit selection* as indicated by the generalized binary variable U_2^*; $\mu(Z_2 \mid v_1^*, U_2^*)$ is the mean of the continuous variable Z_2 in its original scale after both the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the generalized outcome v_1^* and the effect of *explicit selection* as indicated by the generalized binary variable U_2^*; and $\mu(Z_2 \mid v_1^*)$ is the mean of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on Z_1 as indicated by the generalized outcome v_1^*. The generalized form (5.3.67) can be rendered both submodel and outcome specific.

Given the two-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1:

The mean of the continuous variable $\dot{Z}_2(v_1)$ when a correct answer to Item 2 is observed, $\mu(\dot{Z}_2 \mid v_1, u_2 = 1)$, is provided by

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 1) = \sigma(Z_2 \mid v_1) \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)}, \tag{5.3.68}$$

where $\sigma(Z_2 \mid v_1)$, a term cancelled in later derivation, is the standard deviation of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1; $\gamma_2(v_1)$ is the point of dichotomization on the continuous variable $\bar{Z}_2(v_1)$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1, as indicated by the binary outcome v_1; $\phi[\gamma_2(v_1)]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1)$; and $P_2(v_1)$ is the probability of a correct answer to Item 2 given the binary outcome v_1.

The mean of the continuous variable $\dot{Z}_2(v_1)$ when an incorrect answer to Item 2 is observed, $\mu(\dot{Z}_2 \mid v_1, u_2 = 0)$, is provided by

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 0) = - \sigma(Z_2 \mid v_1) \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)}, \tag{5.3.69}$$

where $\sigma(Z_2 \mid v_1)$, a term cancelled in later derivation, is the standard deviation of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on Z_1 as indicated by the binary outcome $v_1; \gamma_2(v_1)$ is the point of dichotomization on $\bar{Z}_2(v_1)$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1; $\phi[\gamma_2(v_1)]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1)$; and $Q_2(v_1)$ is the probability of an incorrect answer to Item 2 given the binary outcome v_1.

Given the three-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1':

The mean of the continuous variable $\dot{Z}_2(v_1')$ when a correct answer to Item 2 is observed, $\mu(\dot{Z}_2 \mid v_1', u_2' = 1)$, is provided by

$$\mu(\dot{Z}_2 \mid v_1', u_2' = 1) = \sigma(Z_2 \mid v_1')(1 - c_2) \frac{\phi[\gamma_2(v_1')]}{P_2'(v_1')}, \tag{5.3.70}$$

where $\sigma(Z_2 \mid v_1')$, a term cancelled in later derivation, is the standard deviation of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1'; c_2 is the coefficient of guessing for Item 2; $\gamma_2(v_1')$ is the point of dichotomization on $\bar{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1'; $\phi[\gamma_2(v_1')]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1')$; and $P_2'(v_1')$ is the probability of a correct answer to Item 2 given the binary outcome v_1'.

The mean of the continuous variable $\dot{Z}_2(v_1')$ when an incorrect answer to Item 2 is observed, $\mu(\dot{Z}_2 \mid v_1', u_2' = 0)$, is provided by

$$\mu(\dot{Z}_2 \mid v_1', u_2' = 0) = - \sigma(Z_2 \mid v_1') \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')}, \tag{5.3.71}$$

where $\sigma(Z_2 \mid v_1')$, a term cancelled in later derivation, is the standard deviation of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1'; $\gamma_2(v_1')$ is the point of dichotomization on $\tilde{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the binary outcome v_1'; $\phi[\gamma_2(v_1')]$ is the density in the standard normal distribution at $\gamma_2(v_1')$; and $Q_2(v_1')$ is the probability of not recognizing the correct alternative to multiple-choice Item 2 given the binary outcome v_1'.

The submodel and outcome specific forms (5.3.68) through (5.3.71) will be used in later derivations. These later derivations will provide an answer to the query, "What is the estimate of ability given the binary score on the second item?"

Given the two-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1:

The variance of the continuous variable $\tilde{Z}_2(v_1)$ when a correct answer to Item 2 is observed, $\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1)$, is provided by

$$\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1) = \frac{\sigma^2(Z_2 \mid v_1, u_2 = 1)}{\sigma^2(Z_2 \mid v_1)}$$

$$= 1 - \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} \left\{ \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} - \gamma_2(v_1) \right\}, \qquad (5.3.72)$$

where $\sigma^2(Z_2 \mid v_1, u_2 = 1)$ is the variance of the continuous variable Z_2 as originally scaled after both the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1 and the effect of *explicit selection* as indicated by u_2, the realization of the binary variable U_2; $\sigma^2(Z_2 \mid v_1)$ is the variance of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1; $\gamma_2(v_1)$ is the point of dichotomization on $\tilde{Z}_2(v_1)$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1; $\phi[\gamma_2(v_1)]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1)$; and $P_2(v_1)$ is the probability of a correct answer to Item 2 subsequent to the outcome v_1.

The variance of the continuous variable $\tilde{Z}_2(v_1)$ when an incorrect answer to Item 2 is observed, $\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 0)$, is provided by

$$\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 0) = \frac{\sigma^2(Z_2 \mid v_1, u_2 = 0)}{\sigma^2(Z_2 \mid v_1)}$$

$$= 1 - \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} \left\{ \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} + \gamma_2(v_1) \right\}, \qquad (5.3.73)$$

where $\sigma^2(Z_2 \mid v_1, u_2 = 0)$ is the variance of the continuous variable Z_2 as originally scaled after both the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1 and the effect of *explicit selection* as indicated by u_2, the realization of the binary variable U_2; $\sigma^2(Z_2 \mid v_1)$ is the variance of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1; $\gamma_2(v_1)$ is the point of dichotomization on $\tilde{Z}_2(v_1)$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1; $\phi[\gamma_2(v_1)]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1)$; and $Q_2(v_1)$ is the probability of an incorrect answer to Item 2 subsequent to the outcome v_1.

Given the three-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1':

The variance of the continuous variable $\tilde{Z}_2(v_1')$, when a correct answer to Item 2 is observed, $\sigma^2(\tilde{Z}_2 \mid v_1', u_2' = 1)$, is provided by:

$$\sigma^2(\bar{Z}_2 \mid v_1',u_2' = 1) = \frac{\sigma^2(Z_2 \mid v_1',u_2' = 1)}{\sigma^2(Z_2 \mid v_1')}$$

$$= 1 - \frac{(1 - c_2)\,\phi\,[\gamma_2(v_1')]}{P_2'(v_1')} \left\{ \frac{(1 - c_2)\,\phi\,[\gamma_2(v_1')]}{P_2'(v_1')} - \gamma_2(v_1') \right\}, \tag{5.3.74}$$

where $\sigma^2(Z_2 \mid v_1',u_2' = 1)$ is the variance of Z_2 as originally scaled after both the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1', and the effect of *explicit selection* as indicated by u_2', the realization of the binary variable U_2'; $\sigma^2(Z_2 \mid v_1')$ is the variance of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1'; c_2 is the coefficient of guessing for Item 2; $\gamma_2(v_1')$ is the point of dichotomization on $\bar{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1'; $\phi[\gamma_2(v_1')]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1')$; and $P_2'(v_1')$ is the probability of a correct answer to Item 2 subsequent to the outcome v_1'.

The variance of the continuous variable $\bar{Z}_2(v_1')$, when an incorrect answer to Item 2 is observed, $\sigma^2(\bar{Z}_2 \mid v_1',u_2' = 0)$, is provided by

$$\sigma^2(\bar{Z}_2 \mid v_1',u_2' = 0) = \frac{\sigma^2(Z_2 \mid v_1',u_2' = 0)}{\sigma^2(Z_2 \mid v_1')}$$

$$= 1 - \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} \left\{ \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} + \gamma_2(v_1') \right\}, \tag{5.3.75}$$

where $\sigma^2(Z_2 \mid v_1',u_2' = 0)$ is the variance of the continuous variable Z_2 as originally scaled after both the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_2 as indicated by the outcome v_1' and the effect of *explicit selection* as indicated by u_2', the realization of the binary variable U_2'; $\sigma(Z_2 \mid v_1')$ is the variance of the continuous variable Z_2 after the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_2 as indicated by the outcome v_1'; $\gamma_2(v_1')$ is the point of dichotomization on $\bar{Z}_2(v_1')$, the standardized form of the continuous variable Z_2 subsequent to the effect of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 as indicated by the outcome v_1'; $\phi[\gamma_2(v_1')]$ is the density in the standard normal distribution evaluated at $\gamma_2(v_1')$; and $Q_2(v_1')$ is the probability of not recognizing the correct alternative to multiple-choice Item 2 subsequent to the outcome v_1'.

The submodel and outcome specific forms (5.3.57) through (5.3.60) and (5.3.72) through (5.3.75) will be used in later derivations. These later derivations will provide answers to the queries, "Which item should be given second?" and "What is the variance of ability given the binary score on this second item?"

Which Item Second? At this point, each item g excluding that item subscripted $g^{(1)}$ is potentially the second item in the tailored test. Which of these $\bar{Z}_g(v_1^*)$ becomes $\bar{Z}_2(v_1^*)$ is then the question. As before, the possible second entries in v_2^* must be probabalistically anticipated in order to obtain the most informative entry. To do this, one rewrites (5.3.55) for the first two equalities as

$$\sigma^2(\Theta^*|v_1^*,U_g^*)[1 - \rho^2(Z_g,\Theta|v_1^*,U_g^*)] = \sigma^2(\Theta^*|v_1^*)[1 - \rho^2(Z_g,\Theta|v_1^*)] \text{ for all } g \text{ where } g\neq g^{(1)}, \tag{5.3.76}$$

which may be rearranged and transposed as

$$\sigma^2(\Theta^* \mid v_1^*,U_g^*) = \sigma^2(\Theta^* \mid v_1^*)[1 - \rho^2(Z_g,\Theta \mid v_1^*]$$

$$+ \sigma^2(\Theta^* \mid v_1^* U_g^*)\rho^2(Z_g,\Theta \mid v_1^*,U_g^*) \text{ for all } g \text{ where } g\neq g^{(1)}. \tag{5.3.77}$$

Also, since any remaining $\bar{Z}_g(v_1^*)$ is potentially $\bar{Z}_2(v_1^*)$ and the second entry in v_2^* must be anticipated, the first two equalities in (5.3.53) may be rewritten as

$$\rho(Z_g,\Theta \mid v_1^*,U_g^*)\,\frac{\sigma(\Theta^* \mid v_1^*,U_g^*)}{\sigma(Z_g \mid v_1^*,U_g^*)} = \rho(Z_g,\Theta \mid v_1^*)\,\frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_g \mid v_1^*)} \text{ for all } g \text{ where } g\neq g^{(1)}. \tag{5.3.78}$$

Upon squaring (5.3.78), one obtains

$$\rho^2(Z_g,\Theta \mid v_1^*,U_g^*)\,\frac{\sigma^2(\Theta^* \mid v_1^*,U_g^*)}{\sigma^2(Z_g \mid v_1^*,U_g^*)} = \rho^2(Z_g,\Theta \mid v_1^*)\,\frac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(Z_g \mid v_1^*)} \quad \text{for all } g \text{ where } g \neq g^{(1)}, \tag{5.3.79}$$

where an explicit solution for $\rho^2(Z_g,\Theta \mid v_1^*,U_g^*)$ yields

$$\rho^2(Z_g,\Theta \mid v_1^*,U_g^*) = \rho^2(Z_g,\Theta \mid v_1^*)\,\frac{\sigma^2(\Theta^* \mid v_1^*)\,\sigma^2(Z_g \mid v_1^*,U_g^*)}{\sigma^2(Z_g \mid v_1^*)\,\sigma^2(\Theta^* \mid v_1^*,U_g^*)} \quad \text{for all } g \text{ where } g \neq g^{(1)}, \tag{5.3.80}$$

which may be substituted into (5.3.77). After this substitution, one has, through rearranging terms,

$$\sigma^2(\Theta^* \mid v_1^*,U_g^*)$$
$$= \sigma^2(\Theta^* \mid v_1^*)\left\{1 - \rho^2(Z_g,\Theta \mid v_1^*)\left[1 - \frac{\sigma^2(Z_g \mid v_1^*,U_g^*)}{\sigma^2(Z_g \mid v_1^*)}\right]\right\} \quad \text{for all } g \text{ where } g \neq g^{(1)} \tag{5.3.81}$$

as the variance of ability given v_1^* for the realizations of U_g^* which can be rendered both submodel and outcome specific.

For the two-parameter normal ogive submodel, where $v_1^* = v_1$ and $U_g^* = U_g$, and given a correct answer to a potential second item g, ($u_g = 1$), a substitution from (5.3.72) into (5.3.81), along with some rearrangement, yields

$$\sigma^2(\Theta^* \mid v_1,u_g = 1)$$
$$= \sigma^2(\Theta^* \mid v_1)\left(1 - \rho^2(Z_g,\Theta \mid v_1)\,\frac{\phi[\gamma_g(v_1)]}{P_g(v_1)}\left\{\frac{\phi[\gamma_g(v_1)]}{P_g(v_1)} - \gamma_g(v_1)\right\}\right) \quad \text{for all } g \text{ where } g \neq g^{(1)} \tag{5.3.82}$$

as the variance of ability given the correct answer. Under the same circumstances and given an incorrect answer to a potential second item g, ($u_g = 0$), a substitution from (5.3.73) into (5.3.81), along with subsequent rearrangement, provides

$$\sigma^2(\Theta^* \mid v_1,u_g = 0)$$
$$= \sigma^2(\Theta^* \mid v_1)\left(1 - \rho^2(Z_g,\Theta \mid v_1)\,\frac{\phi[\gamma_g(v_1)]}{Q_g(v_1)}\left\{\frac{\phi[\gamma_g(v_1)]}{Q_g(v_1)} + \gamma_g(v_1)\right\}\right) \quad \text{for all } g \text{ where } g \neq g^{(1)} \tag{5.3.83}$$

as the variance of ability given the incorrect answer.

For the two-parameter normal ogive submodel, where $v_1^* = v_1$ and $U_g^* = U_g$, substitutions from (5.3.57), (5.3.58), (5.3.82) and (5.3.83) for a potential second item g into (5.2.4) move one closer to an answer to the query: Which item second? After these substitutions, one has

$$\mathscr{E}_{U_g}\sigma^2(\Theta^* \mid v_1,U_g)$$
$$= P_g(v_1)\,\sigma^2(\Theta^* \mid v_1)\left(1 - \rho^2(Z_g,\Theta \mid v_1)\,\frac{\phi[\gamma_g(v_1)]}{P_g(v_1)}\left\{\frac{\phi[\gamma_g(v_1)]}{P_g(v_1)} - \gamma_g(v_1)\right\}\right)$$
$$+ Q_g(v_1)\sigma^2(\Theta^* \mid v_1)\left(1 - \rho^2(Z_g,\Theta \mid v_1)\,\frac{\phi[\gamma_g(v_1)]}{Q_g(v_1)}\left\{\frac{\phi[\gamma_g(v_1)]}{Q_g(v_1)} + \gamma_g(v_1)\right\}\right)$$
$$\text{for all } g \text{ where } g \neq g^{(1)}, \tag{5.3.84}$$

which simplifies using steps identical to those of earlier developments, from (5.3.14) through (5.3.18), to

$$\mathscr{E}_{U_g}\sigma^2(\Theta^* \mid v_1,U_g) = \sigma^2(\Theta^* \mid v_1)\left(1 - \rho^2(Z_g,\Theta \mid v_1)\,\frac{\{\phi[\gamma_g(v_1)]\}^2}{P_g(v_1)Q_g(v_1)}\right) \quad \text{for all } g \text{ where } g \neq g^{(1)}. \tag{5.3.85}$$

The item g, excluding item $g^{(1)}$, for which (5.3.85) is a minimum, is then the most informative second item given the two-parameter normal ogive submodel. Note that the variance of ability, $\sigma^2(\Theta^* \mid v_1)$, is a constant in the evaluation of item g excluding item $g^{(1)}$. Thus the maximum of the second term within the large parentheses in (5.3.85) for the remaining g,

$$\max_{g} \; \rho^2(Z_g,\Theta \mid v_1) \frac{\{\phi[\gamma_g(v_1)]\}^2}{P_g(v_1)Q_g(v_1)} \quad \text{for all } g \text{ where } g \neq g^{(1)}, \tag{5.3.86}$$

will yield an identical g to that provided by the full evaluation of (5.3.85) for the remaining g; and, obviously, the minimum over the remaining g of the reciprocal of the second term within the large parentheses in (5.3.85),

$$\min_{g} \; \frac{P_g(v_1)Q_g(v_1)}{\rho^2(Z_g,\Theta \mid v_1)\{\phi[\gamma_g(v_1)]\}^2} \quad \text{for all } g \text{ where } g \neq g^{(1)}, \tag{5.3.87}$$

will yield the same g as would the full evaluation of (5.3.85) for the remaining items. Either (5.3.86) or (5.3.87) may then be used to choose the most informative second item. While the quantities provided by (5.3.85) yield more information on each remaining item, either (5.3.86) or (5.3.87) provides an identical choice of item. The use of either will introduce a computational savings.

For the three-parameter normal ogive submodel, where $v_1^* = v_1'$ and $U_g^* = U_g'$, and a correct answer to a potential second item g, ($u_g' = 1$), is the anticipated outcome, a substitution from (5.3.74) into (5.3.81), along with some rearrangement, yields

$$\begin{aligned}
&\sigma^2(\Theta^* \mid v_1',u_g' = 1)\\
&\quad = \sigma^2(\Theta^* \mid v_1')\left(1 - \rho^2(Z_g,\Theta \mid v_1')\frac{(1-c_g)\phi[\gamma_g(v_1')]}{P_g'(v_1')}\left\{\frac{(1-c_g)\phi[\gamma_g(v_1')]}{P_g'(v_1')} - \gamma_g(v_1')\right\}\right)\\
&\qquad \text{for all } g \text{ where } g \neq g^{(1)}
\end{aligned} \tag{5.3.88}$$

as the variance of ability given the correct answer. Under the same circumstances and given an incorrect answer to a potential second item g, ($u_g' = 0$), a substitution from (5.3.75) into (5.3.81), along with some rearrangement, provides

$$\begin{aligned}
&\sigma^2(\Theta^* \mid v_1',u_g' = 0)\\
&\quad = \sigma^2(\Theta^* \mid v_1')\left(1 - \rho^2(Z_g,\Theta \mid v_1')\frac{\phi[\gamma_g(v_1')]}{Q_g(v_1')}\left\{\frac{\phi[\gamma_g(v_1')]}{Q_g(v_1')} + \gamma_g(v_1')\right\}\right) \quad \text{for all } g \text{ where } g \neq g^{(1)}
\end{aligned} \tag{5.3.89}$$

as the variance of ability given the incorrect answer.

For the three-parameter normal ogive submodel, where $v_1^* = v_1'$ and $U_g^* = U_g'$, substitutions from (5.3.59), (5.3.60), (5.3.88), and (5.3.89) for a potential second item g into (5.2.4) move one closer to an answer to the query: Which item second? After these substitutions, one has

$$\begin{aligned}
&\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_1',U_g')\\
&\quad = P_g'(v_1')\sigma^2(\Theta^* \mid v_1')\left(1 - \rho^2(Z_g,\Theta \mid v_1')\frac{(1-c_g)\phi[\gamma_g(v_1')]}{P_g'(v_1')}\left\{\frac{(1-c_g)\phi[\gamma_g(v_1')]}{P_g'(v_1')} - \gamma_g(v_1')\right\}\right)\\
&\qquad + Q_g'(v_1')\sigma^2(\Theta^* \mid v_1')\left(1 - \rho^2(Z_g,\Theta \mid v_1')\frac{\phi[\gamma_g(v_1')]}{Q_g(v_1')}\left\{\frac{\phi[\gamma_g(v_1')]}{Q_g(v_1')} + \gamma_g(v_1')\right\}\right)\\
&\qquad \text{for all } g \text{ where } g \neq g^{(1)};
\end{aligned} \tag{5.3.90}$$

which simplifies using steps identical to those of earlier developments, that is from (5.3.23) through (5.3.30), to

$$\begin{aligned}
&\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_1',U_g')\\
&\quad = \sigma^2(\Theta^* \mid v_1')\left(1 - \rho^2(Z_g,\Theta \mid v_1')(1-c_g)\frac{\{\phi[\gamma_g(v_1')]\}^2}{P_g'(v_1')Q_g(v_1')}\right) \quad \text{for all } g \text{ where } g \neq g^{(1)}.
\end{aligned} \tag{5.3.91}$$

In (5.3.91) one now has a computationally convenient solution for the evaluation of each remaining item g. That item g for which (5.3.91) is a minimum is, then, the most informative second item given the three-parameter normal ogive submodel. Again note that the variance of ability, $\sigma^2(\Theta^* \mid v_1')$, is a constant in the evaluation of all the remaining items. Thus the maximum over the remaining items of the second term within the large parentheses in (5.3.91),

$$\max_g \ \rho^2(Z_g,\Theta \mid v_1')(1 - c_g)\frac{\{\phi[\gamma_g(v_1')]\}^2}{P_g'(v_1')Q_g(v_1')} \quad \text{for all } g \text{ where } g \neq g^{(1)}, \tag{5.3.92}$$

will yield an identical g to that provided by the full evaluation of (5.3.91) for the remaining items; and, obviously, the minimum over the remaining g of the reciprocal of the second term within the large parentheses in (5.3.91),

$$\min_g \ \frac{P_g'(v_1')Q_g(v_1')}{\rho^2(Z_g,\Theta \mid v_1')(1 - c_g)\{\phi[\gamma_g(v_1')]\}^2} \quad \text{for all } g \text{ where } g \neq g^{(1)}, \tag{5.3.93}$$

will yield the same g as would the full evaluation of (5.3.91) for the remaining items. Either (5.3.92) or (5.3.93) may then be used to choose the second item. While the quantities provided by (5.3.91) are more informative with respect to each remaining item, either (5.3.92) or (5.3.93) may be used to choose the identical remaining item. The use of either will provide a computational savings.

At this juncture, it may be assumed that the second item has been chosen. In the case of the two-parameter submodel, one may have used either (5.3.85), (5.3.86), or (5.3.87) for this purpose. In the case of the three-parameter normal ogive submodel, one may have used either (5.3.91), (5.3.92), or (5.3.93) for this purpose. This item is subscripted by n within the tailored test. Because this is the second item in the tailored test, its subscript takes on the value 2.

What Is the Estimate of Ability Given the Binary Score on this Second Item? At this point, it is known that: (a) $\bar{Z}_2(v_1^*)$ is in standard score form; (b) $\Theta^*(v_1^*)$ is in its original scale and is distributed about a mean $\mu(\Theta^* \mid v_1^*)$ with a variance $\sigma^2(\Theta^* \mid v_1^*)$; (c) $\bar{Z}_2(v_1^*)$ and $\Theta^*(v_1^*)$ possess, given the updating assumption, a joint bivariate normal distribution; and (d) given this assumption, the property of invariance obtains. Since parallel conditions led to the solutions provided in (2.2.23), (2.2.29), (3.2.15), and (3.2.16), one can now interpret these equations into the present context as estimators of ability:

$$\mu(\Theta^* \mid v_1,u_2 = 1) = \mu(\Theta^* \mid v_1) + \rho(Z_2,\Theta \mid v_1)\sigma(\Theta^* \mid v_1)\mu(\bar{Z}_2 \mid v_1,u_2 = 1) \tag{5.3.94}$$

for a correct answer to Item 2 given the two-parameter normal ogive submodel;

$$\mu(\Theta^* \mid v_1,u_2 = 0) = \mu(\Theta^* \mid v_1) + \rho(Z_2,\Theta \mid v_1)\sigma(\Theta^* \mid v_1)\ \mu(\bar{Z}_2 \mid v_1,u_2 = 0) \tag{5.3.95}$$

for an incorrect answer to Item 2 given the two-parameter normal ogive submodel;

$$\mu(\Theta^* \mid v_1',u_2' = 1) = \mu(\Theta^* \mid v_1') + \rho(Z_2,\Theta \mid v_1')\sigma(\Theta^* \mid v_1')\mu(\bar{Z}_2 \mid v_1',u_2' = 1) \tag{5.3.96}$$

for a correct answer to Item 2 given the three-parameter normal ogive submodel; and

$$\mu(\Theta^* \mid v_1',u_2' = 0) = \mu(\Theta^* \mid v_1') + \rho(Z_2,\Theta \mid v_1')\sigma(\Theta^* \mid v_1')\mu(\bar{Z}_2 \mid v_1',u_2' = 0) \tag{5.3.97}$$

for an incorrect answer to Item 2 given the three-parameter normal ogive submodel.

In this context, then, the generalized expression for the estimator of ability may be written as

$$\mu(\Theta^* \mid v_1^*,U_2^*) = \mu(\Theta^* \mid v_1^*) + \rho(Z_2,\Theta \mid v_1^*)\sigma(\Theta^* \mid v_1^*)\mu(\bar{Z}_2 \mid v_1^*,U_2^*) \tag{5.3.98}$$

where multiplying and dividing the rightmost term by $\sigma(Z_2 \mid v_1^*)$ yield

$$\mu(\Theta^* \mid v_1^*,U_2^*) = \mu(\Theta^* \mid v_1^*) + \rho(Z_2,\Theta \mid v_1^*)\frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)}\sigma(Z_2 \mid v_1^*)\mu(\bar{Z}_2 \mid v_1^*,U_2^*). \tag{5.3.99}$$

But after substituting from (5.3.67) into (5.3.99) one has

$$\mu(\Theta^* \mid v_1^*,U_2^*) = \mu(\Theta^* \mid v_1^*) + \rho(Z_2,\Theta \mid v_1^*)\frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)}\mu(\dot{Z}_2 \mid v_1^*,U_2^*) \tag{5.3.100}$$

where the regression coefficient in the rightmost expression in (5.3.100) now corresponds to that given as the equality in the middle of (5.3.53). After a substitution from the rightmost equality of (5.3.67) into (5.3.100) one obtains, after some rearrangement,

$$\mu(\Theta^* \mid v_1^*,U_2^*) = \mu(\Theta^* \mid v_1^*) - \rho(Z_2,\Theta \mid v_1^*) \frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)} \mu(Z_2 \mid v_1^*)$$

$$+ \rho(Z_2,\Theta \mid v_1^*) \frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)} \mu(Z_2 \mid v_1^*,U_2^*) \qquad (5.3.101)$$

an equation that is reminiscent of the raw score formula in linear regression. In (5.3.101), the first two terms on the right side of the equal sign, after the subtraction is completed, form the intercept. The remaining term on this right side is the weighted mean of Z_2 as originally scaled, given the previous outcome v_1^* and the observed outcome U_2^*. Because of (5.3.101), it is known that for each observed outcome U_2^* given v_1^* that $\mu(\Theta^* \mid v_1^*,U_2^*)$ is the coordinate of $\Theta^*(v_1^*)$ for a point on the line of regression of $\Theta^*(v_1^*)$ on $Z_2(v_1^*)$ where this point's coordinate on $Z_2(v_1^*)$ is given by $\mu(Z_2|v_1^*,U_2^*)$. Remember, $\Theta^*(v_1^*)$ is in the original scale of Θ^*.

Convenient expressions can now be derived for the estimators of ability for the outcome of the encounter between the individual and Item 2 given the previous outcome v_1^*. Under the two-parameter normal ogive submodel, a substitution from (5.3.68) into (5.3.100) provides

$$\mu(\Theta^* \mid v_1,u_2 = 1) = \mu(\Theta^* \mid v_1) + \rho(Z_2,\Theta \mid v_1)\sigma(\Theta^* \mid v_1) \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} \qquad (5.3.102)$$

as the estimator of ability when a correct answer is observed; and a substitution from (5.3.69) into (5.3.100) yields

$$\mu(\Theta^* \mid v_1,u_2 = 0) = \mu(\Theta^* \mid v_1) - \rho(Z_2,\Theta \mid v_1)\sigma(\Theta^* \mid v_1) \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} \qquad (5.3.103)$$

as the estimator of ability when an incorrect answer is observed. Under the three-parameter normal ogive submodel, a substitution from (5.3.70) into (5.3.100) provides

$$\mu(\Theta^* \mid v_1',u_2' = 1) = \mu(\Theta^* \mid v_1') + \rho(Z_2\Theta \mid v_1')\sigma(\Theta^* \mid v_1')\,(1 - c_2) \frac{\phi[\gamma_2(v_1')]}{P_2'(v_1')} \qquad (5.3.104)$$

as the estimator of ability when a correct answer is observed; and a substitution from (5.3.71) into (5.3.100) yields

$$\mu(\Theta^* \mid v_1',u_2' = 0) = \mu(\Theta^* \mid v_1') - \rho(Z_2,\Theta \mid v_1')\sigma(\Theta^* \mid v_1') \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} \qquad (5.3.105)$$

as the estimator of ability when an incorrect answer is observed. In (5.2.102) through (5.3.105) one has expressions that are convenient for the computation of ability for either outcome in the encounter between the individual and Item 2, given both the two- and three-parameter normal ogive submodels.

What Is the Variance of Ability Given the Binary Score on the Second Item? In deciding which item to choose as the second item in the tailored test, it was necessary to obtain solutions for the variances of ability given the realizations of U_g^*. The generalized expression was derived in (5.3.81) and was rendered submodel and outcome specific in (5.3.82), (5.3.83), (5.3.88), and (5.3.89). In this context, it is necessary merely to alias the subscript of the chosen item $g^{(2)}$ by n in each of these equations to obtain the appropriate expressions.

For n equal to 2, the generalized expression for the variance of ability given v_1^* and the realizations of U_2^* is obtained from (5.3.81) through the aliasing of subscripts as

$$\sigma^2(\Theta^* \mid v_1^*,U_2^*) = \sigma^2(\Theta^* \mid v_1^*) \left\{ 1 - \rho^2(Z_2,\Theta \mid v_1^*) \left[1 - \frac{\sigma^2(Z_2 \mid v_1^*,U_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \right] \right\}. \qquad (5.3.106)$$

One makes certain that the items subscripted $g^{(1)}$ and $g^{(2)}$ are distinct items.

For the two-parameter submodel, where $v_1^* = v_1$ and $U_2^* = U_2$, and a correct answer to Item 2, ($u_2 = 1$), is observed the aliasing of the subscripts in (5.3.82) provides

$$\sigma^2(\Theta^* \mid v_1,u_2 = 1) = \sigma^2(\Theta^* \mid v_1) \left(1 - \rho^2(Z_2,\Theta \mid v_1) \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} \left\{ \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} - \gamma_2(v_1) \right\} \right) \qquad (5.3.107)$$

as a convenient expression for the calculation of the variance of ability. For the same circumstances and an incorrect answer to Item 2, ($u_2 = 0$), the aliasing of the subscripts in (5.3.83) yields

$$\sigma^2(\Theta^* \mid v_1,u_2 = 0) = \sigma^2(\Theta^* \mid v_1) \left(1 - \rho^2(Z_2,\Theta \mid v_1) \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} \left\{ \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} + \gamma_2(v_1) \right\} \right) \qquad (5.3.108)$$

as a convenient expression for the calculation of the variance of ability.

For the three-parameter submodel, where $v_1^* = v_1'$ and $U_2^* = U_2'$, and a correct answer to Item 2, $(u_2' = 1)$, is observed the aliasing of the subscripts in (5.3.88) provides

$$\sigma^2(\Theta^* \mid v_1',u_2' = 1)$$

$$= \sigma^2(\Theta^* \mid v_1') \left(1 - \rho^2(Z_2,\Theta \mid v_1') \frac{(1 - c_2)\phi[\gamma_2(v_1')]}{P_2'(v_1')} \left\{ \frac{(1 - c_2)\phi[\gamma_2(v_1')]}{P_2'(v_1')} - \gamma_2(v_1') \right\} \right) \quad (5.3.109)$$

as a convenient expression for the calculation of the variance of ability. For the same circumstances and an incorrect answer to Item 2, $(u_2' = 0)$, the aliasing of the subscripts in (5.3.89) yields

$$\sigma^2(\Theta^* \mid v_1',u_2' = 0) = \sigma^2(\Theta^* \mid v_1') \left(1 - \rho^2(Z_2,\Theta \mid v_1') \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} \left\{ \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} + \gamma_2(v_1') \right\} \right) \quad (5.3.110)$$

as a convenient expression for the calculation of the variance of ability.

Can This Process of Choosing and of Estimating Ability and Its Variance Be Extended to a Third Item? Given the two-parameter submodel, one begins tailoring the second item in an individual's test with $\mu(\Theta^* \mid v_1)$, $\sigma^2(\Theta^* \mid v_1)$ and, consequently, $\sigma(\Theta^* \mid v_1)$ as known values. These values represent the second estimate of this individual's ability, its variance, and its standard deviation. Also a_g^* and b_g^* are known for all g. Given a_g^* and $\sigma(\Theta^* \mid v_1)$, $\rho(Z_g,\Theta \mid v_1)$ can be obtained for the remaining items with subscripts g through the use of (5.3.48). Given a_g^*, b_g^*, $\mu(\Theta^* \mid v_1)$, and $\sigma^2(\Theta^* \mid v_1)$, $\gamma_g(v_1)$ can be obtained for the remaining items with subscript g through the use of (5.3.50); and given $\gamma_g(v_1)$, $P_g(v_1)$ and $Q_g(v_1)$ can be obtained for the remaining items with subscript g through the use of (5.3.57) and (5.3.58), because these remaining items are all eligible to be chosen as the second item. The density at $\gamma_g(v_1)$ is given by

$$\phi[\gamma_g(v_1^*)] = (2\pi)^{-.5} \exp\{-.5[\gamma_g(v_1^*)]^2\} \quad (5.3.111)$$

for the remaining items with subscript g because this relationship obtains by definition. Thus, one can use (5.3.85), or, consequently, (5.3.86) or (5.3.87) to choose the second item, because the required inputs for these equations are known or readily obtainable. What is known or readily obtainable at this juncture is also sufficient for the estimation of ability and the variance of ability. Given the chosen item, $g^{(2)}$, its subscript is aliased by n where n equals 2. Then Item 2 is presented to the individual. If this individual's answer to Item 2 is correct, (5.3.102) is used to estimate this individual's ability; and (5.3.107) is used to estimate the variance of this individual's ability estimate. If this individual's answer to Item 2 is incorrect, (5.3.103) is used to estimate this individual's ability, and (5.3.108) is used to estimate the variance of this individual's ability estimate.

Given the three-parameter submodel, one begins tailoring the second item in an individual's test with $\mu(\Theta^* \mid v_1')$, $\sigma^2(\Theta^* \mid v_1')$, and consequently, $\sigma(\Theta^* \mid v_1')$, as known values. These values represent the second estimate of this individual's ability, its variance, and its standard deviation. Also, a_g^*, b_g^*, and c_g^* are known for all g. Given a_g^* and $\sigma(\Theta^* \mid v_1')$, $\rho(Z_g,\Theta \mid v_1')$ can be obtained for the remaining items with subscript g through the use of (5.3.48). Given a_g^*, b_g^*, $\mu(\Theta^* \mid v_1')$, and $\sigma^2(\Theta^* \mid v_1')$, $\gamma_g(v_1')$ can be obtained for the remaining items with subscript g through the use of (5.3.50); and given $\gamma_g(v_1')$, $P_g'(v_1')$ and $Q_g(v_1')$ can be obtained for the remaining items with subscript g through the use of (5.3.59) and (5.3.65), because these remaining items are all eligible to be chosen as the second item. Notice that $c_g(v_1')$ is merely c_g, as indicated in (5.3.51), when one considers all items eligible to be chosen as the second item. The density at $\gamma_g(v_1')$, $\phi[\gamma_g(v_1')]$, is obtained for the remaining items with subscript g through the use of (5.3.111). Thus one can use (5.3.91) or, consequently, (5.3.92) or (5.3.93) to choose the second item because the required inputs for these equations are known or readily obtainable. What is known or readily obtainable at this juncture is also sufficient for the estimation of ability and the variance of ability. Given the chosen item $g^{(2)}$, its subscript is aliased by n where n equals 2. Then Item 2 is presented to the individual. If this individual's answer is correct, (5.3.104) is used to estimate this individual's ability; and (5.3.109) is used to estimate the variance of this individual's ability estimate. If this individual's answer to Item 2 is incorrect, (5.3.105) is used to estimate this individual's ability; and (5.3.110) is used to estimate the variance of this individual's ability estimate.

The generalized outcome vector v_{n-1}^* or v_2 now contains v_1 and one of four possible entries: $u_{2[g^{(2)}]} = 1$, or $u_{2[g^{(2)}]} = 0$, $u_{2[g^{(2)}]}' = 1$, or $u_{2[g^{(2)}]}' = 0$, where the bracketed notation in the subscript denotes the ability bank subscript of the second chosen item. One now seeks the nth or the third item. The outputs of (5.3.102), (5.3.103), (5.3.104), and (5.3.105) may be rendered in generalized form as $\mu(\Theta^* \mid v_2^*)$; and the outputs of (5.3.107), (5.3.108), (5.3.109), and (5.3.110) may be rendered in generalized form as $\sigma^2(\Theta^* \mid v_2^*)$. At this juncture $\mu(\Theta^* \mid v_2^*)$ and $\sigma^2(\Theta^* \mid v_2^*)$ are known. Of course, also known are a_g^* and b_g^*, in the two-parameter case, and a_g^*, b_g^*, and c_g^* in the three-parameter case, for all the items in the ability bank.

In the following developments it will be established that enough is known for the choice of item, ability estimation, and the estimation of the variance of ability for a tailored test of three or more items. These developments also follow from previous assumptions, those underlying selection theory and the restricted updating procedure.

Under the restricted updating procedure the distribution of $\Theta^*(v_2^*)$, the distribution of Θ^* resulting from *incidental selections* due to *explicit selections* on Z_1 and $\tilde{Z}_2(v_1^*)$, is assumed normal. That this assumption provides a workable approximation was discussed and illustrated earlier. As a consequence, $Z_3(v_2^*)$, the distribution of Z_3 resulting from *incidental selections* due to *explicit selections* on Z_1 and $\tilde{Z}_2(v_1^*)$ is also normal to a workable approximation. The reason for the workability of this approximation is that $Z_3(v_2^*)$ is a linear combination of $\Theta^*(v_2^*)$, the variable of ability, and a homoscedastic random error variable which is normally distributed. The random error variables are assumed under the basic model to be independent from item to item. *Explicit selections* on Z_1 and $\tilde{Z}_2(v_1^*)$ have no effect on the homoscedastic random error in Z_3. As a result, the joint distribution of $Z_3(v_2^*)$ and $\Theta^*(v_2^*)$ is bivariate normal to a workable approximation.

Because bivariate normality is a workable approximation, convenient expressions are now sought for $\rho(Z_3, \Theta \mid v_2^*)$, the correlation between $Z_3(v_2^*)$ and $\Theta^*(v_2^*)$, and $\gamma_3(v_2^*)$, the point of dichotomization on $\tilde{Z}_3(v_2^*)$, or synonomously, the continuous variable $Z_3(v_2^*)$ after its standardization to a mean of zero and variance of unity. These expressions will allow a repetition of the application of the selection or rejection analogy in the tailored testing context. One will then be considering explicit selection or explicit rejection on $\tilde{Z}_3(v_2^*)$, standardized $Z_3(v_2^*)$, where the "cut score" is $\gamma_3(v_2^*)$.

Since the derivations leading to convenient expressions for $\rho(Z_3, \Theta \mid v_2^*)$ and $\gamma_3(v_2^*)$ are of considerable length and detail, only the principal results are presented here. These derivations may be found in Section 5.4. In the course of these derivations, solutions are also found for the parameters $a_3(v_2^*)$, item discriminatory power, and $b_3(v_2^*)$, item difficulty. These parameters are those for Item 3 that are appropriate for $\tilde{\Theta}^*(v_2^*)$, or the continuous variable of ability given the generalized outcome vector v_2^* in standardized form. These parameters provide the property of invariance.

The presented equations apply to any potential third item in the tailored test. At this juncture, there are $(p-2)$ items remaining in the ability bank as possible choices for a third item.

In deriving a convenient expression for $\rho(Z_3, \Theta \mid v_2^*)$, it is necessary to obtain $a_3(v_2^*)$, the item parameter of discriminatory power that is appropriate for $\tilde{\Theta}^*(v_2^*)$, or the standardized form of the continuous variable of ability subsequent to the outcomes recorded in the generalized vector v_2^*. The solution for this parameter is provided by

$$a_3(v_2^*) = a_3^* \, \sigma(\Theta^* \mid v_2^*), \tag{5.3.112}$$

where a_3^* is the parameter of item discriminatory power appropriate for Θ^*, as known; and $\sigma(\Theta^* \mid v_2^*)$ is the standard deviation of Θ^* given the generalized outcome vector v_2^*, as provided by the square root of either (5.3.107) or (5.3.108) in the case of the two-parameter normal ogive submodel, or of either (5.3.109) or (5.3.110) in the case of the three-parameter normal ogive submodel.

A convenient expression for the correlation between Z_3 and Θ^* given the generalized outcome vector v_2^* is provided by

$$\rho(Z_3, \Theta \mid v_2^*) = \frac{a_3(v_2^*)}{\{1 + [a_3(v_2^*)]^2\}^{.5}} = \frac{a_3^* \sigma(\Theta^* \mid v_2^*)}{\{1 + [a_3^* \sigma(\Theta^* \mid v_2^*)]^2\}^{.5}}, \tag{5.3.113}$$

where $a_3(v_2^*)$ is the parameter of item discriminatory power appropriate for $\tilde{\Theta}^*(v_2^*)$, as provided by (5.3.112); a_3^* is the parameter of item discriminatory power appropriate for Θ^*,.as known; and $\sigma(\Theta^* \mid (v_2^*)$ is the standard deviation of Θ^* given the generalized outcome vector v_2^*, as provided by the square root of either (5.3.107) or (5.3.108) in the case of the two-parameter normal ogive submodel, or of either (5.3.109) or (5.3.110) in the case of the three-parameter normal ogive submodel.

In deriving a convenient expression for $\gamma_3(v_2^*)$, it is necessary to obtain a solution for $b_3(v_2^*)$, the parameter of item difficulty appropriate for $\tilde{\Theta}^*(v_2^*)$, or the continuous variable of ability given the generalized outcome vector v_2^* in standardized form. The solution for this parameter is provided by

$$b_3(v_2^*) = \frac{b_3^* - \mu(\Theta^* \mid v_2^*)}{\sigma(\Theta^* \mid v_2^*)}, \tag{5.3.114}$$

where b_3^* is the parameter of item difficulty appropriate for Θ^*, as known; $\mu(\Theta^* \mid v_2^*)$ is the estimator of ability as provided by either (5.3.102) or (5.3.103) in the case of the two-parameter normal ogive submodel, or either (5.3.104) or (5.3.105) in the case of the three-parameter normal ogive submodel; and $\sigma(\Theta^* \mid v_2^*)$ is the standard deviation of Θ^* given the generalized outcome vector v_2^*, as provided by the square root of either (5.3.107) or (5.3.108) in the case of the two-parameter normal ogive submodel, or of either (5.3.109) or (5.3.110) in the case of the three-parameter normal ogive submodel.

A convenient expression for the point of dichotomization on $\tilde{Z}_3(v_2^*)$, the standardized form of the continuous variable Z_3 given the generalized outcome vector v_2^*, is provided by

$$\gamma_3(v_2^*) = \frac{b_3^* - \mu(\Theta^* \mid v_2^*)}{[(a_3^*)^{-2} + \sigma^2(\Theta^* \mid v_2^*)]^{.5}}, \tag{5.3.115}$$

where a_3^* and b_3^* are the parameters of item discriminatory power and item difficulty, respectively, that are appropriate for Θ^*, as known; $\mu(\Theta^* \mid v_2^*)$ is the estimator of ability, as provided by either (5.3.102) or (5.3.103) in the case of the two-parameter normal ogive submodel, or either (5.3.104) or (5.3.105) in the case of the three-parameter normal ogive submodel; and $\sigma^2(\Theta^* \mid v_2^*)$ is the variance of Θ^* given the generalized outcome vector v_2^*, as provided by either (5.3.107) or (5.3.108) in the case of the two-parameter normal ogive submodel, or either (5.3.109) or (5.3.110) in the case of the three-parameter normal ogive submodel.

As indicated by the asterisk on the generalized outcome vector v_2^*, the item parameters defined in (5.3.112) and (5.3.114) are those for both the two- and three-parameter normal ogive submodels. These respective parameters are defined in an identical manner under both submodels. In the case of the three-parameter normal ogive submodel, the third item parameter appropriate for $\tilde{\Theta}^*(v_2^*)$, the standardized form of the variable of ability given the outcome vector v_2^*, is provided by

$$c_3(v_2') = c_3^* = c_3, \tag{5.3.116}$$

which indicates that the lower asymptote of the characteristic curve for Item 3 remains undisturbed by a linear transformation of the continuous variable of ability.

For both submodels, the given item parameters provide the property of invariance. The probability of a correct answer to Item 3 remains invariant for corresponding values of Θ, Θ^*, and $\tilde{\Theta}^*(v_2^*)$. Under these changes in the scale of the continuous variable of ability, the characteristic curve remains invariant.

At this point, it is evident that the process can be extended. This extension of the process consists of choosing the third item and of estimating ability and its variance given the outcome on this chosen item. The extension of this process is developed in Chapter 6 for a tailored test of q_i items where q_i is the subscript n on the last item in the tailored test for individual i.

5.4 Mathematical Proofs

The mathematical formulation that is derived in this section was presented in Section 5.3. The mathematical proofs for this formulation as developed in this section may be omitted by the reader who is seeking a general understanding. The omission of this section will not result in a loss of continuity.

The Correlation Between Z_2 and Θ^* Given the Outcome v_1^*, $\rho(Z_2,\Theta \mid v_1^*)$

In obtaining an explicit solution for $\rho(Z_2,\Theta \mid v_1^*)$, one begins by deriving expressions for $\rho^2(Z_1,Z_2)$ and $\rho^2(Z_1,\Theta)$ for later substitution into (5.3.5). In solving for $\rho^2(Z_1,Z_2)$, an explicit solution for $\sigma(Z_2 \mid v_1^*)$ is obtained from (5.3.1). This solution yields

$$\sigma(Z_2 \mid v_1^*) = \frac{\rho(Z_1,Z_2)\sigma(Z_2)\sigma(Z_1 \mid v_1^*)}{\rho(Z_1,Z_2 \mid v_1^*)\sigma(Z_1)}, \tag{5.4.1}$$

where due to prior scaling $\sigma(Z_1)$ and $\sigma(Z_2)$ are both unity, thus providing

$$\sigma(Z_2 \mid v_1^*) = \frac{\rho(Z_1,Z_2)\sigma(Z_1 \mid v_1^*)}{\rho(Z_1,Z_2 \mid v_1^*)}. \tag{5.4.2}$$

After squaring (5.4.2) and substituting the result into (5.3.3), some rearrangement provides

$$\sigma^2(Z_2)[1-\rho^2(Z_1,Z_2)] = \rho^2(Z_1,Z_2)\sigma^2(Z_1 \mid v_1^*) \left[\frac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)} - 1 \right] \tag{5.4.3}$$

or

$$1-\rho^2(Z_1,Z_2) = \rho^2(Z_1,Z_2)\sigma^2(Z_1 \mid v_1^*) \left[\frac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)} - 1 \right] \tag{5.4.4}$$

because $\sigma^2(Z_2)$ is unity. Dividing both sides of (5.4.4) by $\rho^2(Z_1,Z_2)$, one has

$$\frac{1}{\rho^2(Z_1,Z_2)} - 1 = \sigma^2(Z_1 \mid v_1^*) \left[\frac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)} - 1 \right] \tag{5.4.5}$$

where an explicit solution for $\rho^2(Z_1,Z_2)$ yields

$$\rho^2(Z_1,Z_2) = \left\{ 1+\sigma^2(Z_1 \mid v_1^*) \left[\frac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)} - 1 \right] \right\}^{-1}. \tag{5.4.6}$$

In solving for $\rho^2(Z_1,\Theta)$, one obtains an explicit solution for $\sigma(\Theta^* \mid v_1^*)$ from (5.3.2). This solution yields

$$\sigma(\Theta^* \mid v_1^*) = \frac{\rho(Z_1,\Theta)\sigma(\Theta^*)\sigma(Z_1 \mid v_1^*)}{\rho(Z_1,\Theta \mid v_1^*)\sigma(Z_1)}; \tag{5.4.7}$$

which reduces to

$$\sigma(\Theta^* \mid v_1^*) = \frac{\rho(Z_1,\Theta)\sigma(\Theta^*)\sigma(Z_1 \mid v_1^*)}{\rho(Z_1,\Theta \mid v_1^*)} \tag{5.4.8}$$

because $\sigma(Z_1)$ is unity. After squaring (5.4.8) and substituting the result into (5.3.4), some rearrangement provides

$$\sigma^2(\Theta^*)[1-\rho^2(Z_1,\Theta)] = \rho^2(Z_1,\Theta)\sigma^2(\Theta^*)\sigma^2(Z_1 \mid v_1^*) \left[\frac{1}{\rho^2(Z_1,\Theta \mid v_1^*)} - 1 \right], \tag{5.4.9}$$

where dividing both sides of this equation by the product $\sigma^2(\Theta^*)\rho^2(Z_1,\Theta)$ yields

$$\frac{1}{\rho^2(Z_1,\Theta)} - 1 = \sigma^2(Z_1 \mid v_1^*) \left[\frac{1}{\rho^2(Z_1,\Theta \mid v_1^*)} - 1 \right]. \tag{5.4.10}$$

An explicit solution of (5.4.10) for $\rho^2(Z_1,\Theta)$ now provides

$$\rho^2(Z_1,\Theta) = \left\{ 1+\sigma^2(Z_1 \mid v_1^*) \left[\frac{1}{\rho^2(Z_1,\Theta \mid v_1^*)} - 1 \right] \right\}^{-1}. \tag{5.4.11}$$

After appropriately substituting the results of (5.4.6) and (5.4.11) into (5.3.5), one obtains

$$\frac{\rho(Z_2,\Theta \mid v_1^*) - \rho(Z_1,Z_2 \mid v_1^*)\rho(Z_1,\Theta \mid v_1^*)}{\sqrt{1-\rho^2(Z_1,Z_2 \mid v_1^*)}\sqrt{1-\rho^2(Z_1,\Theta \mid v_1^*)}} \tag{5.4.12}$$

$$= \frac{\rho(Z_2,\Theta) - \left\{ 1+\sigma^2(Z_1 \mid v_1^*)\left[\frac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)} - 1\right]\right\}^{-.5} \left\{ 1+\sigma^2(Z_1 \mid v_1^*)\left[\frac{1}{\rho^2(Z_1,\Theta \mid v_1^*)} - 1\right]\right\}^{-.5}}{\sqrt{1-\left\{1+\sigma^2(Z_1 \mid v_1^*)\left[\frac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)} - 1\right]\right\}^{-1}}\sqrt{1-\left\{1+\sigma^2(Z_1 \mid v_1^*)\left[\frac{1}{\rho^2(Z_1,\Theta \mid v_1^*)} - 1\right]\right\}^{-1}}}$$

where the equality on the right side can be simplified. By placing the terms under the radicals on a common denominator and subtracting terms in the numerator, one has

$$\frac{\rho(Z_2,\Theta)-\left\{1+\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)}-1\right]\right\}^{-.5}\left\{1+\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,\Theta \mid v_1^*)}-1\right]\right\}^{-.5}}{\sqrt{\dfrac{\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)}-1\right]}{1+\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)}-1\right]}}\sqrt{\dfrac{\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,\Theta \mid v_1^*)}-1\right]}{1+\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,\Theta \mid v_1^*)}-1\right]}}} \tag{5.4.13}$$

Multiplication of the numerator and denominator of (5.4.13) by the square roots of the denominators under the radicals, along with some later rearrangement, leads to

$$\frac{\rho(Z_2,\Theta)\left\{1+\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)}-1\right]\right\}^{.5}\left\{1+\sigma^2(Z_1 \mid v_1^*)\left[\dfrac{1}{\rho^2(Z_1,\Theta \mid v_1^*)}-1\right]\right\}^{.5}-1}{\sigma^2(Z_1 \mid v_1^*)\sqrt{\dfrac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)}-1}\sqrt{\dfrac{1}{\rho^2(Z_1,\Theta \mid v_1^*)}-1}}, \tag{5.4.14}$$

where division of the numerator and denominator by $\sigma^2(Z_1 \mid v_1^*)$, along with some later rearrangement, yields

$$\frac{\rho(Z_2,\Theta)\left\{\dfrac{1}{\sigma^2(Z_1 \mid v_1^*)}+\dfrac{1}{\rho^2(Z_1,Z_2 \mid v_1^*)}-1\right\}^{.5}\left\{\dfrac{1}{\sigma^2(Z_1 \mid v_1^*)}+\dfrac{1}{\rho^2(Z_1,\Theta \mid v_1^*)}-1\right\}^{.5}-\dfrac{1}{\sigma^2(Z_1 \mid v_1^*)}}{\left[\dfrac{1}{\rho(Z_1,Z_2 \mid v_1^*)}\right]\sqrt{1-\rho^2(Z_1,Z_2 \mid v_1^*)}\left[\dfrac{1}{\rho(Z_1,\Theta \mid v_1^*)}\right]\sqrt{1-\rho^2(Z_1,\Theta \mid v_1^*)}}. \tag{5.4.15}$$

Multiplication of the numerator and denominator in (5.4.15) by $\rho(Z_1,Z_2 \mid v_1^*)$ and $\rho(Z_1,\Theta \mid v_1^*)$, along with subsequent rearrangement, now leads to

$$\frac{\rho(Z_2,\Theta)\left\{1-\rho^2(Z_1,Z_2 \mid v_1^*)\left[1-\dfrac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}\left\{1-\rho^2(Z_1,\Theta \mid v_1^*)\left[1-\dfrac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}-\dfrac{\rho(Z_1,Z_2 \mid v_1^*)\rho(Z_1,\Theta \mid v_1^*)}{\sigma^2(Z_1 \mid v_1^*)}}{\sqrt{1-\rho^2(Z_1,Z_2 \mid v_1^*)}\sqrt{1-\rho^2(Z_1,Z_2 \mid v_1^*)}}, \tag{5.4.16}$$

which may be substituted back into (5.4.12) for the expression on the right side of the equality. This substitution yields

$$\frac{\rho(Z_2,\Theta \mid v_1^*)-\rho(Z_1,Z_2 \mid v_1^*)\rho(Z_1,\Theta \mid v_1^*)}{\sqrt{1-\rho^2(Z_1,Z_2 \mid v_1^*)}\sqrt{1-\rho^2(Z_1,\Theta \mid v_1^*)}} \tag{5.4.17}$$

$$=\frac{\rho(Z_2,\Theta)\left\{1-\rho^2(Z_1,Z_2 \mid v_1^*)\left[1-\dfrac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}\left\{1-\rho^2(Z_1,\Theta \mid v_1^*)\left[1-\dfrac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}-\dfrac{\rho(Z_1,Z_2 \mid v_1^*)\rho(Z_1,\Theta \mid v_1^*)}{\sigma^2(Z_1 \mid v_1^*)}}{\sqrt{1-\rho^2(Z_1,Z_2 \mid v_1^*)}\sqrt{1-\rho^2(Z_1,\Theta \mid v_1^*)}},$$

where the multiplication of both sides of this equation by their common denominator yields

$$\rho(Z_2,\Theta \mid v_1^*) - \rho(Z_1,Z_2 \mid v_1^*)\rho(Z_1,\Theta \mid v_1^*)$$

$$= \rho(Z_1\Theta)\left\{1-\rho^2(Z_1,Z_2 \mid v_1^*)\left[1-\frac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}\left\{1-\rho^2(Z_1,\Theta \mid v_1^*)\left[1-\frac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}$$

$$-\frac{\rho(Z_1,Z_2 \mid v_1^*)\rho(Z_1,\Theta \mid v_1^*)}{\sigma^2(Z_1 \mid v_1^*)}. \tag{5.4.18}$$

Through transposing and combining expressions in (5.4.18), one obtains

$$\rho(Z_2,\Theta \mid v_1^*) = \rho(Z_2,\Theta)\left\{1-\rho^2(Z_1,Z_2 \mid v_1^*)\left[1-\frac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}\left\{1-\rho^2(Z_1,\Theta \mid v_1^*)\left[1-\frac{1}{\sigma^2(Z_1 \mid v_1^*)}\right]\right\}^{.5}$$

$$+ \rho(Z_1,Z_2 \mid v_1^*)\rho(Z_1,\Theta \mid v_1^*)\left[1-\frac{1}{\sigma^2(Z_1 \mid v_1^*)}\right] \tag{5.4.19}$$

which allows one to proceed to a convenient expression for an explicit solution to $\rho(Z_2,\Theta \mid v_1^*)$.

Solving (5.3.1) and (5.3.2) explicitly for $\rho(Z_1,Z_2 \mid v_1^*)$ and $\rho(Z_1,\Theta \mid v_1^*)$, respectively, it is known that

$$\rho(Z_1,Z_2 \mid v_1^*) = \rho(Z_1,Z_2)\,\frac{\sigma(Z_2)\sigma(Z_1 \mid v_1^*)}{\sigma(Z_1)\sigma(Z_2 \mid v_1^*)} \tag{5.4.20}$$

and

$$\rho(Z_1,\Theta \mid v_1^*) = \rho(Z_1,\Theta)\,\frac{\sigma(\Theta^*)\sigma(Z_1 \mid v_1^*)}{\sigma(Z_1)\sigma(\Theta^* \mid v_1^*)}, \tag{5.4.21}$$

where $\sigma(Z_1)$ and $\sigma(Z_2)$ are both unity. Thus one may write

$$\rho(Z_1,Z_2 \mid v_1^*) = \rho(Z_1,Z_2)\,\frac{\sigma(Z_1 \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)} \tag{5.4.22}$$

and

$$\rho(Z_1,\Theta \mid v_1^*) = \rho(Z_1,\Theta)\sigma(\Theta^*)\,\frac{\sigma(Z_1 \mid v_1^*)}{\sigma(\Theta^* \mid v_1^*)}, \tag{5.4.23}$$

where squared and unsquared substitutions from (5.4.22) and (5.4.23) into (5.4.19), along with later rearrangement, lead to

$$\rho(Z_2,\Theta \mid v_1^*) = \rho(Z_2,\Theta)\sqrt{\left\{1-\frac{\rho^2(Z_1,Z_2)}{\sigma^2(Z_2 \mid v_1^*)}\left[\sigma^2(Z_1 \mid v_1^*)-1\right]\right\}\left\{1-\frac{\rho^2(Z_1,\Theta)\sigma^2(\Theta^*)}{\sigma^2(\Theta^* \mid v_1^*)}\left[\sigma^2(Z_1 \mid v_1^*)-1\right]\right\}}$$

$$+\frac{\rho(Z_1,Z_2)\rho(Z_1\Theta)\sigma(\Theta^*)}{\sigma(Z_2 \mid v_1^*)\sigma(\Theta^* \mid v_1^*)}\left[\sigma^2(Z_1 \mid v_1^*)-1\right]. \tag{5.4.24}$$

Given the basic model, appropriate substitutions from (1.1.21) into (5.4.24) now provide

$$\rho(Z_2,\Theta \mid v_1^*)$$

$$= \rho(Z_2,\Theta)\sqrt{\left\{1-\frac{\rho^2(Z_1,\Theta)\rho^2(Z_2,\Theta)}{\sigma^2(Z_2 \mid v_1^*)}\left[\sigma^2(Z_1 \mid v_1^*)-1\right]\right\}\left\{1-\frac{\rho^2(Z_1,\Theta)\sigma^2(\Theta^*)}{\sigma^2(\Theta^* \mid v_1^*)}\left[\sigma^2(Z_1 \mid v_1^*)-1\right]\right\}}$$

$$+\frac{\rho^2(Z_1,\Theta)\rho(Z_2,\Theta)\sigma(\Theta^*)}{\sigma(Z_2 \mid v_1^*)\sigma(\Theta^* \mid v_1^*)}\left[\sigma^2(Z_1 \mid v_1^*)-1\right], \tag{5.4.25}$$

where one will want to substitute for $\sigma^2(Z_2 \mid v_1^*)$ and $\sigma(Z_2 \mid v_1^*)$. In solving (5.3.3) explicitly for $\sigma^2(Z_2 \mid v_1^*)$, one proceeds by transposition to obtain

$$\sigma^2(Z_2 \mid \nu_1^*) = \sigma^2(Z_2)[1-\rho^2(Z_1,Z_2)] + \rho^2(Z_1,Z_2 \mid \nu_1^*)\sigma^2(Z_2 \mid \nu_1^*), \tag{5.4.26}$$

where $\sigma^2(Z_2)$ is unity. Thus, squaring (5.4.22) and substituting the explicit solution for $\rho^2(Z_1,Z_2 \mid \nu_1^*)$ into (5.4.26) now allows the writing of

$$\sigma^2(Z_2 \mid \nu_1^*) = 1-\rho^2(Z_1,Z_2) + \rho^2(Z_1,Z_2)\sigma^2(Z_1 \mid \nu_1^*); \tag{5.4.27}$$

which may be rewritten as

$$\sigma^2(Z_2 \mid \nu_1^*) = 1+\rho^2(Z_1,\Theta)\rho^2(Z_2,\Theta)[\sigma^2(Z_1 \mid \nu_1^*)-1] \tag{5.4.28}$$

because of (1.1.21). Substituting now from (5.4.28) into (5.4.25), one obtains

$$\begin{aligned}\rho(Z_2,\Theta \mid \nu_1^*) &\\ = \rho(Z_2,\Theta) &\sqrt{\left\{1-\frac{\rho^2(Z_1,\Theta)\rho^2(Z_2,\Theta)[\sigma^2(Z_1 \mid \nu_1^*)-1]}{1+\rho^2(Z_1,\Theta)\rho^2(Z_2,\Theta)[\sigma^2(Z_1 \mid \nu_1^*)-1]}\right\}\left\{1-\frac{\rho^2(Z_1,\Theta)\sigma^2(\Theta^*)}{\sigma^2(\Theta^* \mid \nu_1^*)}[\sigma^2(Z_1 \mid \nu_1^*)-1]\right\}} \\ &+ \frac{\rho^2(Z_1,\Theta)\rho(Z_2,\Theta)\dfrac{\sigma(\Theta^*)}{\sigma(\Theta^* \mid \nu_1^*)}[\sigma^2(Z_1 \mid \nu_1^*)-1]}{\sqrt{1+\rho^2(Z_1,\Theta)\rho^2(Z_2,\Theta)[\sigma^2(Z_1 \mid \nu_1^*)-1]}},\end{aligned} \tag{5.4.29}$$

where placing the terms in the first braces under the first radical to the right of the equal sign over a common denominator, subtracting in the numerator, and multiplying the resulting expression by the other expression in braces under the same radical, yields

$$\begin{aligned}\rho(Z_2,\Theta \mid \nu_1^*) = \rho(Z_2,\Theta)&\sqrt{\frac{1-\rho^2(Z_1,\Theta)\dfrac{\sigma^2(\Theta^*)}{\sigma^2(\Theta^* \mid \nu_1^*)}[\sigma^2(Z_1 \mid \nu_1^*)-1]}{1+\rho^2(Z_1,\Theta)\rho^2(Z_2,\Theta)\,[\sigma^2(Z_1 \mid \nu_1^*)-1]}} \\ &+ \frac{\rho^2(Z_1,\Theta)\rho(Z_2,\Theta)\dfrac{\sigma(\Theta^*)}{\sigma(\Theta^* \mid \nu_1^*)}[\sigma^2(Z_1 \mid \nu_1^*)-1]}{\sqrt{1+\rho^2(Z_1,\Theta)\rho^2(Z_2,\Theta)\,[\sigma^2(Z_1 \mid \nu_1^*)-1]}}.\end{aligned} \tag{5.4.30}$$

In (5.4.30) there are four occurrences of the expression $[\sigma^2(Z_1 \mid \nu_1^*)-1]$. As a result, one will want to solve explicitly for this expression in order to provide further simplification to (5.4.30) through substitution. One begins by rearranging (5.3.4) to obtain

$$\sigma^2(\Theta^* \mid \nu_1^*) = \sigma^2(\Theta^*) - \rho^2(Z_1,\Theta)\sigma^2(\Theta^*) + \rho^2(Z_1,\Theta \mid \nu_1^*)\sigma^2(\Theta^* \mid \nu_1^*) \tag{5.4.31}$$

where squared substitution from (5.4.23) into (5.4.31) allows the writing of

$$\sigma^2(\Theta^* \mid \nu_1^*) = \sigma^2(\Theta^*) - \rho^2(Z_1,\Theta)\sigma^2(\Theta^*) + \rho^2(Z_1,\Theta)\sigma^2(\Theta^*)\sigma^2(Z_1 \mid \nu_1^*). \tag{5.4.32}$$

Some transposing of (5.4.32), along with subsequent rearrangement, leads to

$$\rho^2(Z_1,\Theta)\sigma^2(\Theta^*)[\sigma^2(Z_1 \mid \nu_1^*)-1] = \sigma^2(\Theta^* \mid \nu_1^*)-\sigma^2(\Theta^*), \tag{5.4.33}$$

where division of both sides of the equality by $\rho^2(Z_1,\Theta)\sigma^2(\Theta^*)$ yields

$$[\sigma^2(Z_1 \mid \nu_1^*)-1] = \frac{\sigma^2(\Theta^* \mid \nu_1^*)-\sigma^2(\Theta^*)}{\rho^2(Z_1,\Theta)\sigma^2(\Theta^*)}, \tag{5.4.34}$$

the sought solution. After the four substitutions from (5.4.34) into (5.4.30), the cancellation of several terms and some division lead to

$$\rho(Z_2,\Theta \mid \nu_1^*) = \rho(Z_2,\Theta)\sqrt{\frac{1-\left[1-\dfrac{\sigma^2(\Theta^*)}{\sigma^2(\Theta^* \mid \nu_1^*)}\right]}{1+\rho^2(Z_2,\Theta)\left[\dfrac{\sigma^2(\Theta^* \mid \nu_1^*)}{\sigma^2(\Theta^*)}-1\right]}} + \frac{\rho(Z_2,\Theta)\dfrac{\sigma(\Theta^*)}{\sigma(\Theta^* \mid \nu_1^*)}\left[\dfrac{\sigma^2(\Theta^* \mid \nu_1^*)}{\sigma^2(\Theta^*)}-1\right]}{\sqrt{1+\rho^2(Z_2,\Theta)\left[\dfrac{\sigma^2(\Theta^* \mid \nu_1^*)}{\sigma^2(\Theta^*)}-1\right]}}, \tag{5.4.35}$$

where the numerator under the first radical on the right side of the equal sign in (5.4.35) can be moved from under the radical after subtraction. One then has

$$\rho(Z_2,\Theta \mid v_1^*) = \frac{\rho(Z_2,\Theta)\,\dfrac{\sigma(\Theta^*)}{\sigma(\Theta^* \mid v_1^*)}}{\sqrt{1+\rho^2(Z_2,\Theta)\left[\dfrac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^*)}-1\right]}} + \frac{\rho(Z_2,\Theta)\,\dfrac{\sigma(\Theta^*)}{\sigma(\Theta^* \mid v_1^*)}\left[\dfrac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^*)}-1\right]}{\sqrt{1+\rho^2(Z_2,\Theta)\left[\dfrac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^*)}-1\right]}}, \quad (5.4.36)$$

where the denominators on the right side of the equality are identical, thus permitting the addition of the numerators. After multiplying through the brackets on the second numerator on the right side of the equal sign in (5.4.36), the addition of the numerators yields

$$\rho(Z_2,\Theta \mid v_1^*) = \frac{\rho(Z_2,\Theta)\,\dfrac{\sigma(\Theta^* \mid v_1^*)}{\sigma(\Theta^*)}}{\sqrt{1+\rho^2(Z_2,\Theta)\left[\dfrac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^*)}-1\right]}}, \quad (5.4.37)$$

where unsquared and squared substitutions from (2.3.126) into (5.4.37) allow one to write

$$\rho(Z_2,\Theta \mid v_1^*) = \frac{\dfrac{a_2^*\,\sigma(\Theta^* \mid v_1^*)}{\sqrt{1+[a_2^*\,\sigma(\Theta^*)]^2}}}{\sqrt{1+\dfrac{[a_2^*\,\sigma(\Theta^* \mid v_1^*)]^2 - [a_2^*\,\sigma(\Theta^*)]^2}{1+[a_2^*\,\sigma(\Theta^*)]^2}}}. \quad (5.4.38)$$

After placing the expression under the radical in the lower denominator of (5.4.38) on a common denominator, the subtraction of terms provides

$$\rho(Z_2,\Theta \mid v_1^*) = \frac{\dfrac{a_2^*\,\sigma(\Theta^* \mid v_1^*)}{\sqrt{1+[a_2^*\,\sigma(\Theta^*)]^2}}}{\sqrt{\dfrac{1+[a_2^*\,\sigma(\Theta^* \mid v_1^*]^2}{1+[a_2^*\,\sigma(\Theta^*)]^2}}} \quad (5.4.39)$$

where multiplication of the numerator and denominator by their common denominator yields the result

$$\rho(Z_2,\Theta \mid v_1^*) = \frac{a_2^*\,\sigma(\Theta^* \mid v_1^*)}{\sqrt{1+[a_2^*\,\sigma(\Theta^* \mid v_1^*)]^2}}. \quad (5.4.40)$$

By definition, one has

$$\rho(Z_2,\Theta \mid v_1^*) = \frac{a_2(v_1^*)}{\sqrt{1+[a_2(v_1^*)]^2}} \quad (5.4.41)$$

where $a_2(v_1^*)$ is appropriate for $\Theta^*(v_1^*)$ after its standardization to a mean of zero and a variance of unity. Upon squaring (5.4.41), one obtains

$$\rho^2(Z_2,\Theta \mid v_1^*) = \frac{[a_2(v_1^*)]^2}{1+[a_2(v_1^*)]^2}, \quad (5.4.42)$$

where the substitution of the squared result of (5.4.40), after transposition, yields

$$\frac{[a_2(v_1^*)]^2}{1+[a_2(v_1^*)]^2} = \frac{[a_2^*\,\sigma(\Theta^* \mid v_1^*)]^2}{1+[a_2^*\,\sigma(\Theta^* \mid v_1^*)]^2}. \quad (5.4.43)$$

Upon clearing the denominators in (5.4.43), one has

$$[a_2(v_1^*)]^2 + [a_2(v_1^*)]^2\,[a_2^*\,\sigma(\Theta^* \mid v_1^*)]^2 = [a_2^*\,\sigma(\Theta^* \mid v_1^*)]^2 + [a_2(v_1^*)]^2\,[a_2^*\,\sigma(\Theta^* \mid v_1^*)]^2, \quad (5.4.44)$$

where subtraction of the common expressions on both sides of the equality yields

$$[a_2(v_1^*)]^2 = [a_2^* \sigma(\Theta^* \mid v_1^*)]^2. \tag{5.4.45}$$

Taking the square root of (5.4.45) provides

$$a_2(v_1^*) = a_2^* \sigma(\Theta^* \mid v_1^*), \tag{5.4.46}$$

indicating that $a_2(v_1^*)$, appropriate for $\tilde{\Theta}^*(v_1^*)$ or $\Theta^*(v_1^*)$ after its standardization to a mean of zero and variance of unity, is obtained by the multiplication of two knowns, a_2^* and $\sigma(\Theta^* \mid v_1^*)$. Thus (5.4.40) provides a convenient expression for $\rho(Z_2, \Theta \mid v_1^*)$, which was one of the two solutions sought.

The Point of Dichotomization on Standardized Z_2 Given the Outcome v_1^*, $\gamma_2(v_1^*)$

The other solution one now seeks is a convenient expression for $\gamma_2(v_1^*)$, the point of dichotomization on $\tilde{Z}_2(v_1^*)$, or, synonomously, $Z_2(v_1^*)$ after its standardization to a mean of zero and a variance of unity. In obtaining this solution, one first defines the rescaling of Θ^* for which the parameter $a_2(v_1^*)$ is appropriate. This definition will lead to a solution for $b_2(v_1^*)$. For completeness, $c_2(v_1^*)$ will then be defined. At this juncture, both $\rho(Z_2, \Theta \mid v_1^*)$ and $b_2(v_1^*)$ will be known. Consequently, $\gamma_2(v_1^*)$ will be known, because $\gamma_2(v_1^*)$ is by definition merely the product of $\rho(Z_2, \Theta \mid v_1^*)$ and $b_2(v_1^*)$. After obtaining this solution, the important property of invariance will be evaluated.

After *incidental selection* on Θ^* due to *explicit selection* on Z_1, Θ^* can be standardized to a mean of zero and a variance of unity. This standardization is accomplished in the usual manner through

$$\tilde{\theta}^*(v_1^*) = \frac{\theta^* - \mu(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_1^*)}, \tag{5.4.47}$$

where $\tilde{\Theta}^*(v_1^*)$ is the continuous variable of ability for which the parameter $a_2(v_1^*)$ is appropriate. The appropriate parameter $b_2(v_1^*)$ is obtained through

$$b_2(v_1^*) = \frac{b_2^* - \mu(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_1^*)}, \tag{5.4.48}$$

where the expression on the right side of the equality resembles the similarly positioned expression in (5.4.47). This resemblance occurs because, consistent with the definition of the difficulty parameter, $b_2(v_1^*)$ is rendered in the standard scale of ability, in this case that of $\tilde{\Theta}^*(v_1^*)$. The parameters $a_2(v_1^*)$ and $b_2(v_1^*)$ are, as usual, defined identically in both submodels. For the three-parameter normal ogive submodel, one has

$$c_2(v_1^*) = c_2^* = c_2 \tag{5.4.49}$$

again, as in the case of (3.3.65), because a change in the scale of ability leaves the lower asymptote of the regression of a binary, multiple-choice item on ability unchanged.

Since $\gamma_2(v_1^*)$, the point of dichotomization on $\tilde{Z}_2(v_1^*)$, is by definition

$$\gamma_2(v_1^*) = \rho(Z_2, \Theta \mid v_1^*)\, b_2(v_1^*), \tag{5.4.50}$$

substitutions from (5.4.40) and (5.4.48) into (5.4.50), along with a cancellation of terms and divisions of the numerator and denominator by a_2^*, now yields

$$\gamma_2(v_1^*) = \frac{b_2^* - \mu(\Theta^* \mid v_1^*)}{[(a_2^*)^{-2} + \sigma^2(\Theta^* \mid v_1^*)]^{.5}}, \tag{5.4.51}$$

the solution that was sought. In (5.4.51) one has a convenient expression for the point of dichotomization on $\tilde{Z}_2(v_1^*)$ where the required inputs are known.

The Property of Invariance

Remember, the property of invariance pertains to the probability of passing item g. This probability must remain invariant for any particular level, rather than numerical value, of the continuous variable of ability under changes in its scale. Invariance is obtained by simple transformations of the item parameters given a change in the scale of the continuous variable of ability. Earlier, in connection (2.3.122), the property of invariance was discussed in connection with an arbitrary prescription for the scale of the continuous variable of ability. In this context, the discussion will be extended to include a change in the scale of the continuous variable of ability resulting from the standardization of Θ^* subsequent to the *incidental selection* on Θ^* imposed by *explicit selection* on Z_1. Extending equation (2.3.122), invariance requires the result

$$\gamma_2(\theta) = \gamma_2(\theta^*) = \gamma_2[\tilde{\theta}^*(\boldsymbol{v}_1^*)], \tag{5.4.52}$$

where the second item is potentially any one of the remaining $(p-1)$ unused items in the ability bank. The point of dichotomization $\gamma_2[\tilde{\theta}^*(\boldsymbol{v}_1^*)]$ would be the lower limit of integration for the standardized conditional distribution of $\tilde{Z}_2(\boldsymbol{v}_1^*)$ given $\tilde{\theta}^*(\boldsymbol{v}_1^*)$, where the integral yields the probability of either producing or recognizing a correct answer to the second item in the tailored test. In this context, $\gamma_2[\tilde{\theta}^*(\boldsymbol{v}_1^*)]$ operates as did $\gamma_2(\theta)$ in the context of (2.1.15). When the level of ability is arbitrarily fixed, $\gamma_2[\tilde{\theta}^*(\boldsymbol{v}_1^*)]$ must equal $\gamma_2(\theta)$ even though the particular numerical values $\tilde{\theta}^*(\boldsymbol{v}_1^*)$ and θ are unequal because of intervening transformations on the scale of the continuous variable of ability. The equality yields an invariant probability of producing or recognizing a correct answer at the arbitrarily fixed level of ability. Thus the probabilities yielding the item characteristic curve remain undisturbed when the equality in (5.4.52) is maintained. Given invariance, $\gamma_2[\tilde{\theta}^*(\boldsymbol{v}_1^*)]$ is defined by

$$\gamma_2[\tilde{\theta}^*(\boldsymbol{v}_1^*)] = -a_2(\boldsymbol{v}_1^*)\,[\tilde{\theta}^*(\boldsymbol{v}_1^*) - b_2(\boldsymbol{v}_1^*)] \tag{5.4.53}$$

under the necessary condition of equality asserted in (5.4.52). Substitutions from (2.1.15), (2.3.121), and (5.4.53) into (5.4.52) yield

$$-a_2(\theta - b_2) = -a_2^*(\theta^* - b_2^*) = -a_2(\boldsymbol{v}_1^*)\,[\tilde{\theta}^*(\boldsymbol{v}_1^*) - b_2(\boldsymbol{v}_1^*)], \tag{5.4.54}$$

where substitutions from (5.4.46), (5.4.47), and (5.4.48) into the rightmost member of the equality allow one to write

$$-a_2(\theta - b_2) = -a_2^*(\theta^* - b_2^*) = -a_2^*(\theta^* - b_2^*). \tag{5.4.55}$$

Subsequent substitutions from (2.3.118), (2.3.119), and (2.3.120) into (5.4.55) provide proof of the necessary condition

$$-a_2(\theta - b_2) = -a_2(\theta - b_2) = -a_2(\theta - b_2), \tag{5.4.56}$$

which demonstrates the property of invariance and the appropriateness of the parameters $a_2(\boldsymbol{v}_1^*)$ and $b_2(\boldsymbol{v}_1^*)$ for transformed ability $\tilde{\Theta}^*(\boldsymbol{v}_1^*)$.

Some Consequences of the Submodels and the Updating Assumption

The Probabilities for the Realizations of U_2^* Given the Generalized Outcome $\boldsymbol{v}_1^*$. In earlier developments *explicit selection* on the continuous variable Z_g was considered when Z_g was in standard score form and γ_g was the "cut score" or point of dichotomization on Z_g. Here one is considering *explicit selection* on the standardized continuous variable $\tilde{Z}_2(\boldsymbol{v}_1^*)$. The standardized values of variable $\tilde{Z}_2(\boldsymbol{v}_1^*)$ are defined by

$$\tilde{\zeta}_2(\boldsymbol{v}_1^*) = \frac{\zeta_2 - \mu(Z_2 \mid \boldsymbol{v}_1^*)}{\sigma(Z_2 \mid \boldsymbol{v}_1^*)}, \tag{5.4.57}$$

where $\tilde{Z}_2(\boldsymbol{v}_1^*)$, like Z_g, because of the updating assumption, is distributed bivariate normally with respect to the continuous variable of ability, scaled as either Θ, Θ^*, $\Theta^*(\boldsymbol{v}_1^*)$, or $\tilde{\Theta}^*(\boldsymbol{v}_1^*)$.

Under the two-parameter normal ogive submodel and the updating assumption:

The probability of a correct answer to Item 2 given the outcome $\boldsymbol{v}_1$, $\Pr(u_2 = 1 \mid \boldsymbol{v}_1)$, is designated as $P_2(\boldsymbol{v}_1)$. This probability by definition is

$$\Pr(u_2 = 1 \mid \boldsymbol{v}_1) = P_2(\boldsymbol{v}_1) = \int_{\gamma_2(\boldsymbol{v}_1)}^{\infty} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(\boldsymbol{v}_1), \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*(\boldsymbol{v}_1)\, d\tilde{\zeta}_2(\boldsymbol{v}_1) \tag{5.4.58}$$

an equation which may be viewed as resulting from the substitutions of $\gamma_2(\boldsymbol{v}_1)$ for γ_g, of $\tilde{\zeta}_2(\boldsymbol{v}_1)$ for ζ_g, and of $\tilde{\theta}^*(\boldsymbol{v}_1)$ for θ into (2.3.26). Since $\gamma_2(\boldsymbol{v}_1)$, $\tilde{\zeta}_2(\boldsymbol{v}_1)$, $\tilde{\theta}^*(\boldsymbol{v}_1)$, and, consequently, $P_2(\boldsymbol{v}_1)$

have respective definitions that correspond to those of γ_g, ζ_g, θ, and P_g in this earlier context, the appropriate substitution of $\gamma_2(\nu_1)$ and $\bar{\zeta}_2(\nu_1)$ into the earlier solution for P_g, (2.3.31), provides

$$\Pr(u_g = 1 \mid \nu_1) = P_2(\nu_1) = \int_{\gamma_2(\nu_1)}^{\infty} (2\pi)^{-.5} \exp\{-.5[\bar{\zeta}_2(\nu_1)]^2\}\, d\bar{\zeta}_2(\nu_1)$$

$$= \int_{-\infty}^{-\gamma_2(\nu_1)} (2\pi)^{-.5} \exp\{-.5[\bar{\zeta}_2(\nu_1)]^2\}\, d\bar{\zeta}_g(\nu_1)$$

$$= \Phi[-\gamma_2(\nu_1)] \tag{5.4.59}$$

as the probability of a correct answer given these circumstances. In (5.4.59), $\Phi[-\gamma_2(\nu_1)]$ is the cumulative normal distribution function evaluated for the interval on $\bar{Z}_2(\nu_1)$ extending from negative infinity to negative $\gamma_2(\nu_1)$.

The probability of an incorrect answer to Item 2, given the outcome ν_1, $\Pr(u_2 = 0 \mid \nu_1)$, is designated as $Q_2(\nu_1)$. This probability by definition is

$$\Pr(u_2 = 0 \mid \nu_1) = Q_2(\nu_1) = \int_{-\infty}^{\gamma_2(\nu_1)} \int_{-\infty}^{\infty} \phi[\bar{\zeta}_2(\nu_1), \bar{\theta}^*(\nu_1)]\, d\bar{\theta}^*(\nu_1)\, d\bar{\zeta}_2(\nu_1), \tag{5.4.60}$$

an equation which may be viewed as resulting from the substitutions of $\gamma_2(\nu_1)$ for γ_g, of $\bar{\zeta}_2(\nu_1)$ for ζ_g, and of $\bar{\theta}^*(\nu_1)$ for θ into (2.3.32). Since $\gamma_2(\nu_1)$, $\bar{\zeta}_2(\nu_1)$, $\bar{\theta}^*(\nu_1)$, and, consequently, $Q_2(\nu_1)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, and Q_g in this earlier context, the appropriate substitutions of $\gamma_2(\nu_1)$ and $\bar{\zeta}_2(\nu_1)$ into the earlier solution for Q_g, (2.3.35), provides

$$\Pr(u_2 = 0 \mid \nu_1) = Q_2(\nu_1) = \int_{-\infty}^{\gamma_2(\nu_1)} (2\pi)^{-.5} \exp\{-.5[\bar{\zeta}_2(\nu_1)]^2\}\, d\bar{\zeta}_2(\nu_1) = \Phi[\gamma_2(\nu_1)] \tag{5.4.61}$$

as the probability of an incorrect answer to Item 2 given these circumstances. In (5.4.61), $\Phi[\gamma_2(\nu_1)]$ is the cumulative normal distribution function evaluated for the interval on $\bar{Z}_2(\nu_1)$ extending from negative infinity to $\gamma_2(\nu_1)$.

Under the three-parameter normal ogive submodel and the updating assumption:

The probability of a correct answer to Item 2 given the outcome ν_1', $\Pr(u_2' = 1 \mid \nu_1')$, is designated as $P_2'(\nu_1')$. This probability by definition is

$$\Pr(u_2' = 1 \mid \nu_1') = P_2'(\nu_1')$$

$$= \int_{\gamma_2(\nu_1')}^{\infty} \int_{-\infty}^{\infty} \phi[\bar{\zeta}_2(\nu_1'), \bar{\theta}^*(\nu_1')]\, d\bar{\theta}^*(\nu_1')\, d\bar{\zeta}_2(\nu_1')$$

$$+ c_2 \int_{-\infty}^{\gamma_2(\nu_1')} \int_{-\infty}^{\infty} \phi[\bar{\zeta}_2(\nu_1'), \bar{\theta}^*(\nu_1')]\, d\bar{\theta}^*(\nu_1')\, d\bar{\zeta}_2(\nu_1') \tag{5.4.62}$$

an equation which may be viewed as resulting from the substitution of $\gamma_2(\nu_1')$ for γ_g, of $\bar{\zeta}_2(\nu_1')$ for ζ_g, of $\bar{\theta}^*(\nu_1')$ for θ, and of c_2 for c_g into (3.3.1). Since $\gamma_2(\nu_1')$, $\bar{\zeta}_2(\nu_1')$ $\bar{\theta}^*(\nu_1')$, c_2, and, consequently, $P_2'(\nu_1')$ have respective definitions that correspond to those of γ_g, ζ_g, θ, c_g and P_g' in this earlier context, the appropriate substitutions of $\gamma_2(\nu_1')$, $\bar{\zeta}_2(\nu_1')$, and c_2 into the earlier solution for P_g', (3.3.5), provides

$$\Pr(u_2' = 1 \mid \nu_1') = P_2'(\nu_1')$$

$$= c_2 + (1 - c_2) \int_{\gamma_2(\nu_1')}^{\infty} (2\pi)^{-.5} \exp\{-.5[\bar{\zeta}_2(\nu_1')]^2\}\, d\bar{\zeta}_2(\nu_1')$$

$$= c_2 + (1 - c_2) \int_{-\infty}^{-\gamma_2(\nu_1')} (2\pi)^{-.5} \exp\{-.5[\bar{\zeta}_2(\nu_1')]^2\}\, d\bar{\zeta}_2(\nu_1')$$

$$= c_2 + (1 - c_2)\, \Phi[-\gamma_2(\nu_1')] \tag{5.4.63}$$

as the probability of a correct answer given these circumstances. In (5.4.63), $\Phi[-\gamma_2(v_1')]$ is the cumulative normal distribution function evaluated for the interval on $\bar{Z}_2(v_1')$ extending from negative infinity to negative $\gamma_2(v_1')$.

The probability of an incorrect answer to Item 2 given the outcome v_1', $\Pr(u_2' = 0 \mid v_1')$, is designated as $Q_2'(v_1')$. This probability by definition is

$$\Pr(u_2' = 0 \mid v_1') = Q_2'(v_1') = (1 - c_2)\int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi[\bar{\zeta}_2(v_1'),\bar{\theta}^*(v_1')]\, d\bar{\theta}^*(v_1')\, d\bar{\zeta}_2(v_1') \quad (5.4.64)$$

an equation which may be viewed as resulting from the substitutions of $\gamma_2(v_1')$ for γ_g, of $\bar{\zeta}_2(v_1')$ for ζ_g, of $\bar{\theta}^*(v_1')$ for θ, and of c_2 for c_g into (3.3.6). Since $\gamma_2(v_1')$, $\bar{\zeta}_2(v_1')$, $\bar{\theta}^*(v_1')$, c_2, and, consequently, $Q_2'(v_1')$ have respective definitions that correspond to those of γ_g, ζ_g, θ, c_g, and Q_g' in this earlier context, the appropriate substitutions of $\gamma_2(v_1')$, $\bar{\zeta}_2(v_1')$, and c_2 into the earlier solution for Q_g', (3.3.8), provides

$$\begin{aligned}\Pr(u_2' = 0 \mid v_1') &= Q_2'(v_1') \\ &= (1 - c_2)\int_{-\infty}^{\gamma_2(v_1')} (2\pi)^{-.5} \exp\{-.5[\bar{\zeta}_2(v_1')]^2\}\, d\bar{\zeta}_2(v_1') \\ &= (1 - c_2)\, \Phi[\gamma_2(v_1')] \end{aligned}\quad (5.4.65)$$

as the probability of an incorrect answer to Item 2 given these circumstances. In (5.4.65), $\Phi[\gamma_2(v_1')]$ is the cumulative normal distribution function evaluated for the interval of $\bar{Z}_2(v_1')$ extending from negative infinity to $\gamma_2(v_1')$.

The Means of $\bar{Z}_2(v_1^*)$ and $\dot{Z}(v_1^*)$ for the Realizations of U_2^*. At this juncture, the $\bar{Z}_2(v_1^*)$ are standardized variables which, by assumption, and like Z_g in a previous setting, are distributed bivariate normally with respect to the continuous variable of ability. The previous developments obtained under identical circumstances can thus be interpreted into the present context to provide the solutions for the means of the $\bar{Z}_2(v_1^*)$. In order to obtain the solutions for the means of the continuous variables $\dot{Z}_2(v_1^*)$ whose mean deviate values, $\dot{\zeta}_2(v_1^*)$, are defined by

$$\dot{\zeta}_2(v_1^*) = \zeta_2 - \mu(Z_2 \mid v_1^*), \quad (5.4.66)$$

some further development is required.

Given the two-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1:

The mean of $\bar{Z}_2(v_1)$ when a correct answer to Item 2 is observed, $\mu(\bar{Z}_2 \mid v_1, u_2 = 1)$, is by definition

$$\begin{aligned}\mu(\bar{Z}_2 \mid v_1, u_2 = 1) &= \mathscr{E}(\bar{Z}_2 \mid v_1, u_2 = 1) \\ &= \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \bar{\zeta}_2(v_1)\, \phi^*[\bar{\zeta}_2(v_1),\bar{\theta}^*(v_1)]\, d\bar{\theta}^*(v_1)\, d\bar{\zeta}_2(v_1)\end{aligned}\quad (5.4.67)$$

where, in this context, the joint density function $\phi^*[\bar{\zeta}_2(v_1),\bar{\theta}^*(v_1)]$ is given by

$$\phi^*[\bar{\zeta}_2(v_1),\bar{\theta}^*(v_1)] = \frac{\phi[\bar{\zeta}_2(v_1),\bar{\theta}^*(v_1)]}{P_2(v_1)}. \quad (5.4.68)$$

Equations (5.4.67) and (5.4.68) may be viewed as resulting from equations (2.3.36) and (2.3.37) after substitutions of $\gamma_2(v_1)$ for γ_g, $\bar{\zeta}_2(v_1)$ for ζ_g, $\bar{\theta}^*(v_1)$ for θ, and $P_2(v_1)$ for P_g. Since $\gamma_2(v_1)$, $\bar{\zeta}_2(v_1)$, $\bar{\theta}^*(v_1)$, $P_2(v_1)$, and, consequently, $\mu(\bar{Z}_2 \mid v_1, u_2 = 1)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, P_g, and $\mu(Z_g \mid u_g = 1)$ in this earlier context, one can then interpret the earlier solution for $\mu(Z_g \mid u_g = 1)$, (2.3.48), into the present context as

$$\mu(\bar{Z}_2 \mid v_1, u_2 = 1) = \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} \quad (5.4.69)$$

which provides the mean of $\tilde{Z}_2(v_1)$ when a correct answer to a free-response Item 2 is observed.

Notice that (5.4.67) may be rewritten as

$$\mu(\tilde{Z}_2 \mid v_1,u_2 = 1) = \frac{1}{\sigma(Z_2 \mid v_1)} \mu(\dot{Z}_2 \mid v_1,u_2 = 1)$$

$$= \frac{1}{\sigma(Z_2 \mid v_1)} \mathscr{E}(\dot{Z}_2 \mid v_1,u_2 = 1)$$

$$= \frac{1}{\sigma(Z_2 \mid v_1)} \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \dot{\zeta}_2(v_1)\ \phi^*[\tilde{\zeta}_2(v_1),\tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1) \quad (5.4.70)$$

because of (5.4.57) and (5.4.66). In the equality involving a double integral in (5.4.70), a constant with respect to the integration, the reciprocal of $\sigma(Z_2 \mid v_1)$, has been moved outside of this integral. This operation renders the expected value obtained through integration from that of $\mu(\tilde{Z}_2 \mid v_1,u_2 = 1)$ in (5.4.67) to that of $\mu(\dot{Z}_2 \mid v_1,u_2 = 1)$ in (5.4.70). Because of (5.4.69) and (5.4.70), one may then write

$$\mu(\dot{Z}_2 \mid v_1,u_2 = 1) = \sigma(Z_2 \mid v_1)\mu(\tilde{Z}_2 \mid v_1,u_2 = 1) = \sigma(Z_2 \mid v_1)\ \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} \quad (5.4.71)$$

as the mean of $\dot{Z}_2(v_1)$ when a correct answer to Item 2 is observed. But from (5.4.66) and (5.4.70) it is known that

$$\mu(\dot{Z}_2 \mid v_1,u_2 = 1) = \mathscr{E}(\dot{Z}_2 \mid v_1,u_2 = 1)$$

$$= \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid v_1)]\ \phi^*[\tilde{\zeta}_2(v_1),\tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1) \quad (5.4.72)$$

which involves obtaining the expected value of a continuous variable in mean deviate form. Equation (5.4.72) may also be written as

$$\mu(\dot{Z}_2 \mid v_1,u_2 = 1)$$

$$= \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \zeta_2\ \phi^*[\tilde{\zeta}_2(v_1),\tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1)$$

$$- \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \mu(Z_2 \mid v_1)\ \phi^*[\tilde{\zeta}_2(v_1),\tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1) \quad (5.4.73)$$

an equation involving a difference between double integrals. In (5.4.73) the double integral where the integrand is a product of ζ_2 is by definition $\mathscr{E}(Z_2 \mid v_1,u_2 = 1)$ or $\mu(Z_2 \mid v_1,u_2 = 1)$. Thus, one may write

$$\mu(\dot{Z}_2 \mid v_1,u_2 = 1) = \mu(Z_2 \mid v_1,u_2 = 1) \quad (5.4.74)$$

$$- \mu(Z_2 \mid v_1) \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \phi^*[\tilde{\zeta}_2(v_1),\tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1)$$

where the constant with respect to the integration, $\mu(Z_2 \mid v_1)$, has been written outside of the double integral. After a substitution from (5.4.68) into (5.4.74), one has

$$\mu(\dot{Z}_2 \mid v_1,u_2 = 1) = \mu(Z_2 \mid v_1,u_2 = 1)$$

$$- \frac{\mu(Z_2 \mid v_1)}{P_2(v_1)} \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1),\tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1) \quad (5.4.75)$$

or merely

$$\mu(\dot{Z}_2 \mid v_1,u_2 = 1) = \mu(Z_2 \mid v_1,u_2 = 1) - \mu(Z_2 \mid v_1) \quad (5.4.76)$$

because of (5.4.58). In (5.4.76), $\mu(Z_2 \mid v_1,u_2 = 1)$ is the mean of the continuous variable Z_2 as originally scaled subsequent to the effects of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 and of *explicit selection* on the continuous variable Z_2 given the earlier *incidental selection;* and $\mu(Z_2 \mid v_1)$ is the mean of the continuous variable Z_2 given the *incidental selection* resulting from *explicit selection* on the continuous variable Z_1.

The mean of $\tilde{Z}_2(v_1)$ when an incorrect answer to Item 2 is observed, $\mu(\tilde{Z}_2 \mid v_1, u_2 = 0)$, is by definition

$$\mu(\tilde{Z}_2 \mid v_1, u_2 = 0) = \mathscr{E}(\tilde{Z}_2 \mid v_1, u_2 = 0)$$
$$= \int_{-\infty}^{\gamma_2(v_1)} \int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1)\ \phi^{\dagger}[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1), \quad (5.4.77)$$

where, in this context, the joint density function, $\phi^{\dagger}[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]$, is given by

$$\phi^{\dagger}[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] = \frac{\phi[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]}{Q_2(v_1)}. \quad (5.4.78)$$

Equations (5.4.77) and (5.4.78) may be viewed as resulting from equations (2.3.49) and (2.3.50) after substitutions of $\gamma_2(v_1)$ for γ_g, $\tilde{\zeta}_2(v_1)$ for ζ_g, $\tilde{\theta}^*(v_1)$ for θ, and $Q_2(v_1)$ for Q_g. Since $\gamma_2(v_1)$, $\tilde{\zeta}_2(v_1)$, $\tilde{\theta}^*(v_1)$, $Q_2(v_1)$, and, consequently, $\mu(\tilde{Z}_2 \mid v_1, u_2 = 0)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, Q_g, and $\mu(Z_g \mid u_g = 0)$ in this earlier context, one can then interpret the earlier solution for $\mu(Z_g \mid u_g = 0)$, (2.3.59), into the present context as

$$\mu(\tilde{Z}_2 \mid v_1, u_2 = 0) = -\frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} \quad (5.4.79)$$

which provides the mean of $\tilde{Z}_2(v_1)$ when an incorrect answer to free-response Item 2 is observed.

Notice that (5.4.77) may be rewritten as

$$\mu(\tilde{Z}_2 \mid v_1, u_2 = 0) = \frac{1}{\sigma(Z_2 \mid v_1)}\ \mu(\dot{Z}_2 \mid v_1, u_2 = 0)$$
$$= \frac{1}{\sigma(Z_2 \mid v_1)}\ \mathscr{E}(\dot{Z}_2 \mid v_1, u_2 = 0)$$
$$= \frac{1}{\sigma(Z_2 \mid v_1)} \int_{-\infty}^{\gamma_2(v_1)} \int_{-\infty}^{\infty} \dot{\zeta}_2(v_1)\ \phi^{\dagger}[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1) \quad (5.4.80)$$

because of (5.4.57) and (5.4.66). This operation renders the expected value obtained through integration from that of $\mu(\tilde{Z}_2 \mid v_1, u_2 = 0)$ in (5.4.77) to that of $\mu(\dot{Z}_2 \mid v_1, u_2 = 0)$ in (5.4.80). Because of (5.4.79) and (5.4.80), one may write

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 0) = \sigma(Z_2 \mid v_1)\mu(\tilde{Z}_2 \mid v_1, u_2 = 0) = -\sigma(Z_2 \mid v_1)\ \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} \quad (5.4.81)$$

as the mean of $\dot{Z}_2(v_1)$ when an incorrect answer to Item 2 is observed. But from (5.4.66) and (5.4.80) it is known that

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 0) = \mathscr{E}(\dot{Z}_2 \mid v_1, u_2 = 0)$$
$$= \int_{-\infty}^{\gamma_2(v_1)} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid v_1)]\ \phi^{\dagger}[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1), \quad (5.4.82)$$

which involves obtaining the expected value of a continous variable in mean deviate form. Equation (5.4.82) may also be written as

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 0)$$
$$= \int_{-\infty}^{\gamma_2(v_1)} \int_{-\infty}^{\infty} \zeta_2\ \phi^{\dagger}[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1)$$
$$- \int_{-\infty}^{\gamma_2(v_1)} \int_{-\infty}^{\infty} \mu(Z_2 \mid v_1)\ \phi^{\dagger}[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]\ d\tilde{\theta}^*(v_1)\ d\tilde{\zeta}_2(v_1), \quad (5.4.83)$$

an equation involving a difference between double integrals. In (5.4.83), the double integral where the integrand is a product of ζ_2 is by definition $\mathscr{E}(Z_2 \mid v_1, u_2 = 0)$, or $\mu(Z_2 \mid v_1, u_2 = 0)$. Thus, one may write

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 0) = \mu(Z_2 \mid v_1, u_2 = 0)$$

$$- \mu(Z_2 \mid v_1) \int_{-\infty}^{\gamma_2(v_1)} \int_{-\infty}^{\infty} \phi^{\dagger}[\zeta_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\zeta_2(v_1), \qquad (5.4.84)$$

where the constant with respect to the integration, $\mu(Z_2 \mid v_1)$, has been written outside of the double integral. After a substitution from (5.4.78) into (5.4.84), one has

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 0) = \mu(Z_2 \mid v_1, u_2 = 0)$$

$$- \frac{\mu(Z_2 \mid v_1)}{Q_2(v_1)} \int_{-\infty}^{\gamma_2(v_1)} \int_{-\infty}^{\infty} \phi[\zeta_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\zeta_2(v_1), \qquad (5.4.85)$$

or merely

$$\mu(\dot{Z}_2 \mid v_1, u_2 = 0) = \mu(Z_2 \mid v_1, u_2 = 0) - \mu(Z_2 \mid v_1), \qquad (5.4.86)$$

because of (5.4.60). In (5.4.86), $\mu(Z_2 \mid v_1, u_2 = 0)$ is the mean of the continuous variable Z_2 as originally scaled subsequent to the effects of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 and of *explicit selection* on the continuous variable Z_2 given the earlier *incidental selection;* and $\mu(Z_2 \mid v_1)$ is the mean of the continuous variable Z_2 given the *incidental selection* resulting from *explicit selection* on the continuous variable Z_1.

Given the three-parameter normal ogive submodel, the updating assumption, and the previous outcome

The mean of $\tilde{Z}_2(v_1')$ when a correct answer to Item 2 is observed, $\mu(\tilde{Z}_2 \mid v_1', u_2' = 1)$, is by definition

$$\mu(\tilde{Z}_2 \mid v_1', u_2' = 1) = \mathscr{E}(\tilde{Z}_2 \mid v_1', u_2' = 1)$$

$$= \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1') \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1') \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'), \qquad (5.4.87)$$

where the joint density function $\phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')]$ is given by

$$\phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] = \frac{\phi[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')]}{P_2'(v_1')}. \qquad (5.4.88)$$

Equations (5.4.87) and (5.4.88) may be viewed as resulting from equations (3.3.11) and (3.3.12) after substitutions of $\gamma_2(v_1')$ for γ_g, $\tilde{\zeta}_2(v_1')$ for ζ_g, $\tilde{\theta}^*(v_1')$ for θ, $P_2'(v_1')$ for P_g', and c_2 for c_g. Since $\gamma_2(v_1')$, $\tilde{\zeta}_2(v_1')$, $\tilde{\theta}^*(v_1')$, $P_2'(v_1')$, c_2, and, consequently, $\mu(\tilde{Z}_2 \mid v_1', u_2' = 1)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, P_g', c_g, and $\mu(Z_g \mid u_g' = 1)$ in this earlier context, one can then interpret the earlier solution for $\mu(Z_g \mid u_g' = 1)$, (3.3.21), into the present context as

$$\mu(\tilde{Z}_2 \mid v_1', u_2' = 1) = (1 - c_2) \frac{\phi[\gamma_2(v_1')]}{P_2'(v_1')}, \qquad (5.4.89)$$

which provides the mean of $\tilde{Z}_2(v_1')$ when a correct answer to a multiple-choice Item 2 is observed.

Notice that (5.4.87) may be written as

$$\mu(\tilde{Z}_2 \mid v_1',u_2' = 1) = \frac{1}{\sigma(Z_2 \mid v_1')}\,\mu(\dot{Z}_2 \mid v_1',u_2' = 1) = \frac{1}{\sigma(Z_2 \mid v_1')}\,\mathscr{E}(\dot{Z}_2 \mid v_1',u_2' = 1)$$

$$= \frac{1}{\sigma(Z_2 \mid v_1')}\left\{\int_{\gamma_2(v_1')}^{\infty}\int_{-\infty}^{\infty}\dot{\zeta}_2(v_1')\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')\right.$$

$$\left. + c_2\int_{-\infty}^{\gamma_2(v_1')}\int_{-\infty}^{\infty}\dot{\zeta}_2(v_1')\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')\right\}, \qquad (5.4.90)$$

because of (5.4.57) and (5.4.66). In the equality involving the two double integrals in (5.4.90), a constant with respect to the integration, the reciprocal of $\sigma(Z_2 \mid v_1')$, has been moved outside of these double integrals. This operation renders the expected value obtained through the composite of double integrals from that of $\mu(\dot{Z}_2 \mid v_1',u_2' = 1)$ in (5.4.87) to that of $\mu(\tilde{Z}_2 \mid v_1',u_2' = 1)$ in (5.4.90). Because of (5.4.89) and (5.4.90) one may then write

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 1) = \sigma(Z_2 \mid v_1')\mu(\tilde{Z}_2 \mid v_1',u_2' = 1) = \sigma(Z_2 \mid v_1')(1 - c_2)\,\frac{\phi[\gamma_2(v_1')]}{P_2'(v_1')} \qquad (5.4.91)$$

as the mean of $\dot{Z}_2(v_1')$ when a correct answer to Item 2 is observed. But from (5.4.66) and (5.4.90), it is known that

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 1) = \mathscr{E}(\dot{Z}_2 \mid v_1',u_2' = 1)$$

$$= \int_{\gamma_2(v_1')}^{\infty}\int_{-\infty}^{\infty}[\zeta_2 - \mu(Z_2 \mid v_1')]\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')$$

$$+ c_2\int_{-\infty}^{\gamma_2(v_1')}\int_{-\infty}^{\infty}[\zeta_2 - \mu(Z_2 \mid v_1')]\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1'),$$

$$(5.4.92)$$

which involves the taking of an expected value of a continuous variable in mean deviate form. Equation (5.4.92) may also be written as

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 1)$$

$$= \int_{\gamma_2(v_1')}^{\infty}\int_{-\infty}^{\infty}\zeta_2\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')$$

$$- \int_{\gamma_2(v_1')}^{\infty}\int_{-\infty}^{\infty}\mu\,(Z_2 \mid v_1')\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1^*)]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')$$

$$+ c_g\int_{-\infty}^{\gamma_2(v_1')}\int_{-\infty}^{\infty}\zeta_2\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')$$

$$- c_g\int_{-\infty}^{\gamma_2(v_1')}\int_{-\infty}^{\infty}\mu(Z_2 \mid v_1')\,\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\,\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1'), \qquad (5.4.93)$$

an equation involving four double integrals. In (5.4.93), the sum of the two double integrals where the integrand is a product of ζ_2 is by definition $\mathscr{E}(Z_2 \mid v_1',u_2' = 1)$ or $\mu(Z_2 \mid v_1',u_1' = 1)$. Thus, one may write

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 1) = \mu(Z_2 \mid v_1',u_2' = 1)$$

$$-\mu(Z_2 \mid v_1')\int_{\gamma_2(v_1')}^{\infty}\int_{-\infty}^{\infty}\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')$$

$$-c_2\mu(Z_2 \mid v_1')\int_{-\infty}^{\gamma_2(v_1')}\int_{-\infty}^{\infty}\phi^{*\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1'), \qquad (5.4.94)$$

where the constant with respect to the integration, $\mu(Z_2 \mid v_1')$, has been written outside of the double integrals. After a substitution from (5.4.88) into (5.4.94), one has, after some rearrangement,

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 1)$$

$$= \frac{\mu(Z_2 \mid v_1')}{P_2'(v_1')} \left\{ \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right.$$

$$\left. + \, c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1'), \, \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right\}, \tag{5.4.95}$$

which may be written as

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 1) = \mu(Z_2 \mid v_1',u_2' = 1) - \mu(Z_2 \mid v_1'), \tag{5.4.96}$$

after a substitution from (5.4.62) for the expression within the braces in (5.4.95). In (5.4.96), $\mu(Z_2 \mid v_1',u_2' = 1)$ is the mean of the continuous variable Z_2 as originally scaled subsequent to the effects of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 and of *explicit selection* on the continuous variable Z_2 given the earlier *incidental selection;* and $\mu(Z_2 \mid v_1')$ is the mean of Z_2 given the *incidental selection* resulting from *explicit selection* on the continuous variable Z_1.

The mean of $\tilde{Z}_2(v_1')$ when an incorrect answer to Item 2 is observed, $\mu(\tilde{Z}_2 \mid v_1',u_2' = 0)$, is by definition

$$\mu(\tilde{Z}_2 \mid v_1',u_2' = 0) = \mathscr{E}(\tilde{Z}_2 \mid v_1',u_2' = 0)$$

$$= (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1') \, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'), \tag{5.4.97}$$

where, in this context, the joint density function, $\phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]$, is given by

$$\phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')] = \frac{\phi[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]}{Q_2'(v'_1)}. \tag{5.4.98}$$

Equations (5.4.97) and (5.4.98) may be viewed as resulting from equations (3.3.22) and (3.3.23) after substitutions of $\gamma_2(v_1')$ for γ_g, $\tilde{\zeta}_2(v_1')$ for ζ_g, $\tilde{\theta}^*(v_1')$ for θ, $Q_2'(v_1')$ for Q_g', and c_2 for c_g. Since $\gamma_2(v_1')$, $\tilde{\zeta}_2(v_1')$, $\tilde{\theta}^*(v_1')$, $Q_2'(v_1')$, $Q_2(v_1')$, c_2, and, consequently, $\mu(\tilde{Z}_2 \mid v_1',u_2' = 0)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, Q_g', Q_g, c_g, and $\mu(Z_g \mid u_2' = 0)$ in this earlier context, one can then interpret the earlier solution for $\mu(Z_g \mid u_g' = 0)$, (3.3.31), into the present context as

$$\mu(\tilde{Z}_2 \mid v_1',u_2' = 0) = -\frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')}, \tag{5.4.99}$$

which provides the mean of $\tilde{Z}_2(v_1')$ when an incorrect answer to a multiple-choice Item 2 is observed.

Notice that (5.4.97) may be written as

$$\mu(\tilde{Z}_2 \mid v_1',u_2' = 0)$$

$$= \frac{1}{\sigma(Z_2 \mid v_1')} \mu(\dot{Z}_2 \mid v_1',u_2' = 0)$$

$$= \frac{1}{\sigma(Z_2 \mid v_1')} \mathscr{E}(\dot{Z}_2 \mid v_1',u_2' = 0)$$

$$= \frac{(1 - c_2)}{\sigma(Z_2 \mid v_1')} \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \dot{\zeta}(v_1') \, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \tag{5.4.100}$$

because of (5.4.57) and (5.4.66). In the equality involving the double integral, a constant with respect to the integration, the reciprocal of $\sigma(Z_2 \mid v_1')$, has been moved outside the double integral. This operation renders the expected value obtained as the product of $(1 - c_2)$ and the double integral from that of $\mu(\tilde{Z}_2 \mid v_1',u_2' = 0)$ in (5.4.97) to that of $\mu(\dot{Z}_2 \mid v_1',u_2' = 0)$ in (5.4.100). Because of (5.4.99) and (5.4.100), one may then write

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 0) = \sigma(Z_2 \mid v_1')\mu(\tilde{Z}_2 \mid v_1',u_2' = 0) = -\sigma(Z_2 \mid v_1') \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} \quad (5.4.101)$$

as the mean of $\dot{Z}_2(v_1')$ when an incorrect answer to Item 2 is observed. But from (5.4.66) and (5.4.100) it is known that

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 0) = \mathscr{E}(\dot{Z}_2 \mid v_1',u_2' = 0)$$
$$= (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid v_1')]\, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1'),$$
$$(5.4.102)$$

which involves obtaining an expected value of a continuous variable in mean deviate form. Equation (5.4.102) may also be written as

$$\mu(\dot{Z} \mid v_1',u_2' = 0)$$
$$= (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2\, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')\,]d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1')$$
$$- (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \mu(Z_2 \mid v_1')\, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1'), \quad (5.4.103)$$

an equation involving the difference between the products of a constant, $(1 - c_2)$, and a double integral. In (5.4.103), the product of this constant and the double integral where the integrand is a product of ζ_2 is by definition $\mathscr{E}(Z_2 \mid v_1',u_2' = 0)$ or $\mu(Z_2 \mid v_1',u_2' = 0)$. Thus one may write

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 0) = \mu(Z_2 \mid v_1',u_2' = 0)$$
$$- \mu(Z_2 \mid v_1')\,(1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1'), \quad (5.4.104)$$

where the constant with respect to the integration $\mu(Z_2 \mid v_1')$, has been written outside of the double integral. After a substitution from (5.4.98) into (5.4.104), one has

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 0) = \mu(Z_2 \mid v_1',u_2' = 0)$$
$$-\mu(Z_2 \mid v_1') \frac{(1 - c_2)}{Q_2'(v_1')} \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1'),\tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1') \quad (5.4.105)$$

or merely

$$\mu(\dot{Z}_2 \mid v_1',u_2' = 0) = \mu(Z_2 \mid v_1',u_2' = 0) - \mu(Z_2 \mid v_1'), \quad (5.4.106)$$

because of (5.4.64). In (5.4.106), $\mu(Z_2 \mid v_1',u_2' = 0)$ is the mean of the continuous variable Z_2 as originally scaled subsequent to the effects of *incidental selection* resulting from *explicit selection* on the continuous variable Z_1 and of *explicit selection* on the continuous variable Z_2 given the earlier *incidental selection;* and $\mu(Z_2 \mid v_1')$ is the mean of the continuous variable Z_2 given the *incidental selection* resulting from *explicit selection* on the continuous variable Z_1.

A generalized expression may now be written for $\mu(\dot{Z}_2 \mid v_1',U_2^*)$. As can be seen in (5.4.76), (5.4.86), (5.4.96), and (5.4.106), this expression is given by

$$\mu(\dot{Z}_2 \mid v_1^*,U_2^*) = \mu(Z_2 \mid v_1^*,U_2^*) - \mu(Z_2 \mid v_1^*), \quad (5.4.107)$$

a form that proved useful in an earlier context. In this earlier context, ability estimation was compared to a more familiar method of estimation. This method was that of linear regression.

The Variances of $\tilde{Z}_2(v_1^*)$ for the Realizations of U_2^*. By assumption it is known that the standardized continuous variables $\tilde{Z}_2(v_1^*)$, like Z_g in an earlier setting, are distributed bivariate normally with respect to the continuous variable of ability. Thus, the earlier developments obtained under identical circumstances can be interpreted into the present context to provide solutions for the variances of the $\tilde{Z}_2(v_1^*)$ for each of the realizations of U_2^*. However, some further developments are required. These developments demonstrate that the variances of $\tilde{Z}_2(v_1^*)$ for each of the realizations of U_2^* possess identities that consist of the ratios of variances. These ratios of variances are required in the derivations which provide solutions for the variances of ability given the outcomes v_1^* and U_2^*.

Given the two-parameter normal ogive submodel, the updating assumption, and the previous outcome v_1:

The variance of $\tilde{Z}_2(v_1')$ when a correct answer to free-response Item 2 is observed, $\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1)$, is by definition

$$\begin{aligned}\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1) &= \mathscr{E}(\tilde{Z}_2^2 \mid v_1, u_2 = 1) - \mu^2(\tilde{Z}_2 \mid v_1, u_2 = 1)\\ &= \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} [\tilde{\zeta}_2(v_1)]^2 \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1')\\ &\quad - \left\{ \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1) \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1) \right\}^2, \end{aligned} \tag{5.4.108}$$

where the joint density function $\phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)]$ is provided by (5.4.68). Equations (5.4.68) and (5.4.108) may be viewed as resulting from (2.3.36), (2.3.37), and (2.3.62) after substitutions of $\gamma_2(v_1)$ for γ_g, $\tilde{\zeta}_2(v_1)$ for ζ_g, $\tilde{\theta}^*(v_1)$ for θ, and $P_2(v_1)$ for P_g. Since $\gamma_2(v_1)$, $\tilde{\zeta}_2(v_1)$, $\tilde{\theta}^*(v_1)$, $P_2(v_1)$, and, consequently, $\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, P_g, and $\sigma^2(Z_g, u_g = 1)$ in this earlier context, one can then interpret the earlier solution for $\sigma^2(Z_g \mid u_g = 1)$, (2.3.82), into the present context as

$$\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1) = 1 - \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} \left\{ \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} - \gamma_2(v_1) \right\}, \tag{5.4.109}$$

which provides the variance of $\tilde{Z}_2(v_1)$ when a correct answer to a free-response Item 2 is observed.

Because of (5.4.57), (5.4.66), and (5.4.70), an abbreviated form of (5.4.108) may be written as

$$\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1) = \frac{1}{\sigma^2(Z_2 \mid v_1)} [\mathscr{E}(\dot{Z}_2^2 \mid v_1, u_2 = 1) - \mu^2(\dot{Z}_2 \mid v_1, u_2 = 1)], \tag{5.4.110}$$

where the expression in brackets is a variance. This expression by definition is the variance of $\dot{Z}_2(v_1)$ when a correct answer to Item 2 is observed. The equation for this variance then is

$$\sigma^2(\dot{Z}_2 \mid v_1, u_2 = 1) = \mathscr{E}(\dot{Z}_2^2 \mid v_1, u_2 = 1) - \mu^2(\dot{Z}_2 \mid v_1, u_2 = 1). \tag{5.4.111}$$

A substitution from (5.4.111) into (5.4.110) allows one to write

$$\sigma^2(\tilde{Z}_2 \mid v_1, u_2 = 1) = \frac{\sigma^2(\dot{Z}_2 \mid v_1, u_2 = 1)}{\sigma^2(Z_2 \mid v_1)}, \tag{5.4.112}$$

where $\sigma^2(\dot{Z}_2 \mid v_1, u_2 = 1)$ has an identity which after substitution into (5.4.112) provides a more useful expression.

In solving for this identity, it is necessary to examine further the expected value $\mathscr{E}(\dot{Z}_2 \mid v_1, u_2 = 1)$. Because of (5.4.66), this expected value of squared mean deviates may be expanded as

$$\begin{aligned}\mathscr{E}(\dot{Z}_2^2 \mid v_1, u_2 = 1) &= \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} [\dot{\zeta}(v_1)]^2 \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1)\\ &= \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid v_1)]^2 \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1)\\ &= \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1)\\ &\quad - 2\mu(Z_2 \mid v_1) \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \zeta_2 \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1)\\ &\quad + \mu^2(Z_2 \mid v_1) \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1), \end{aligned} \tag{5.4.113}$$

where the constant, $\mu(Z_2 \mid v_1)$ or $\mu^2(Z_2 \mid v_1)$, has been moved outside of two of the double integrals. In the last equality in (5.4.113), the first double integral defines $\mathscr{E}(Z_2^2 \mid v_1,u_2 = 1)$, where one then has

$$\mathscr{E}(Z_2^2 \mid v_1,u_2 = 1) = \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1) \tag{5.4.114}$$

and the second double integral defines $\mu(Z_2 \mid v_1,u_2 = 1)$, where one has

$$\mu(Z_2 \mid v_1,u_2 = 1) = \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \zeta_2 \, \phi^*[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1). \tag{5.4.115}$$

Substituting from (5.4.68), (5.4.114), and (5.4.115) into (5.4.113), one may write

$$\begin{aligned} \mathscr{E}(\dot{Z}_2^2 \mid v_1,u_2 = 1) &= \mathscr{E}(Z_2^2 \mid v_1,u_2 = 1) \\ &\quad - 2\mu(Z_2 \mid v_1) \, \mu(Z_2 \mid v_1,u_2 = 1) \\ &\quad + \frac{\mu^2(Z_2 \mid v_1)}{P_2(v_1)} \int_{\gamma_2(v_1)}^{\infty} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1), \tilde{\theta}^*(v_1)] \, d\tilde{\theta}^*(v_1) \, d\tilde{\zeta}_2(v_1) \end{aligned} \tag{5.4.116}$$

where this equation may be simplified to

$$\begin{aligned} \mathscr{E}(\dot{Z}_2^2 \mid v_1,u_2 = 1) &= \mathscr{E}(Z_2^2 \mid v_1,u_2 = 1) \\ &\quad - 2\mu(Z_2 \mid v_1) \, \mu(Z_2 \mid v_1,u_2 = 1) + \mu^2(Z_2 \mid v_1) \end{aligned} \tag{5.4.117}$$

because of (5.4.58). A substitution from (5.4.117) into (5.4.111) now provides

$$\begin{aligned} \sigma^2(\dot{Z}_2 \mid v_1,u_2 = 1) &= \mathscr{E}(\dot{Z}_2^2 \mid v_1,u_2 = 1) \\ &\quad - 2\mu(Z_2 \mid v_1) \, \mu(Z_2 \mid v_1,u_2 = 1) \\ &\quad + \mu^2(Z_2 \mid v_1) - \mu^2(\dot{Z}_2 \mid v_1,u_2 = 1) \end{aligned} \tag{5.4.118}$$

where a further substitution from the squared result of (5.4.76) yields

$$\begin{aligned} \sigma^2(\dot{Z}_2 \mid v_1,u_2 = 1) &= \mathscr{E}(\dot{Z}_2^2 \mid v_1,u_2 = 1) \\ &\quad - 2\mu(Z_2 \mid v_1) \, \mu(Z_2 \mid v_1,u_2 = 1) \\ &\quad + \mu^2(Z_2 \mid v_1) - \mu^2(Z_2 \mid v_1,u_2 = 1) \\ &\quad + 2\mu(Z_2 \mid v_1) \, \mu(Z_2 \mid v_1,u_2 = 1) - \mu^2(Z_2 \mid v_1). \end{aligned} \tag{5.4.119}$$

The cancellation of terms in (5.4.119) provides

$$\sigma^2(\dot{Z}_2 \mid v_1,u_2 = 1) = \mathscr{E}(\dot{Z}_2^2 \mid v_1,u_2 = 1) - \mu^2(Z_2 \mid v_1,u_2 = 1). \tag{5.4.120}$$

But, by definition, one has

$$\sigma^2(Z_2 \mid v_1,u_2 = 1) = \mathscr{E}(Z_2^2 \mid v_1,u_2 = 1) - \mu^2(Z_2 \mid v_1,u_2 = 1), \tag{5.4.121}$$

where a substitution from (5.4.121) into (5.4.120) provides the sought identity

$$\sigma^2(\dot{Z}_2 \mid v_1,u_2 = 1) = \sigma^2(Z_2 \mid v_1,u_2 = 1). \tag{5.4.122}$$

The substitution of this identity of $\sigma^2(\dot{Z}_2 \mid v_1,u_2 = 1)$ into (5.4.112) yields

$$\sigma^2(\tilde{Z}_2 \mid v_1,u_2 = 1) = \frac{\sigma^2(Z_2 \mid v_1,u_2 = 1)}{\sigma^2(Z_2 \mid v_1)}, \tag{5.4.123}$$

which because of (5.4.109) may be extended to provide the useful form

$$\begin{aligned} \sigma^2(\tilde{Z}_2 \mid v_1,u_2 = 1) &= \frac{\sigma^2(Z_2 \mid v_1,u_2 = 1)}{\sigma^2(Z_2 \mid v_1)} \\ &= 1 - \frac{\phi(\gamma_2(v_1))}{P_2(v_1)} \left\{ \frac{\phi[\gamma_2(v_1)]}{P_2(v_1)} - \gamma_2(v_1) \right\}, \end{aligned} \tag{5.4.124}$$

which allows a substitution into a development from selection theory. This substitution will lead to a convenient expression for the variance of ability given the outcome $\boldsymbol{v}_1$ when a correct answer to free-response Item 2 is observed.

The variance of $\tilde{Z}_2(\boldsymbol{v}_1)$ when an incorrect answer to free-response Item 2 is observed, $\sigma^2(\tilde{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0)$, is by definition

$$
\begin{aligned}
\sigma^2(\tilde{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0) &= \mathscr{E}(\tilde{Z}_2^2 \mid \boldsymbol{v}_1,u_2 = 0) - \mu^2(\tilde{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0) \\
&= \int_{-\infty}^{\gamma_2(\boldsymbol{v}_1)} \int_{-\infty}^{\infty} [\tilde{\zeta}_2(\boldsymbol{v}_1)]^2\, \phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*(\boldsymbol{v}_1)\, d\tilde{\zeta}_2(\boldsymbol{v}_1) \\
&\quad - \left\{ \int_{-\infty}^{\gamma_2(\boldsymbol{v}_1)} \int_{-\infty}^{\infty} \tilde{\zeta}_2(\boldsymbol{v}_1)\, \phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*(\boldsymbol{v}_1)\, d\tilde{\zeta}_2(\boldsymbol{v}_1) \right\}^2
\end{aligned}
\qquad (5.4.125)
$$

where the joint density function $\phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]$ is provided by (5.4.78). Equations (5.4.78) and (5.4.125) may be viewed as resulting from (2.3.49), (2.3.50), and (2.3.85) after substitutions of $\gamma_2(\boldsymbol{v}_1)$ for γ_g, $\tilde{\zeta}_2(\boldsymbol{v}_1)$ for ζ_g, $\tilde{\theta}^*(\boldsymbol{v}_1)$ for θ, and $Q_2(\boldsymbol{v}_1)$ for Q_g. Since $\gamma_2(\boldsymbol{v}_1)$, $\tilde{\zeta}_2(\boldsymbol{v}_1)$, $\tilde{\theta}^*(\boldsymbol{v}_1)$, $Q_2(\boldsymbol{v}_1)$, and, consequently, $\sigma^2(\tilde{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, Q_g, and $\sigma^2(Z_g \mid u_g = 0)$ in this earlier context, one can then interpret the earlier solution for $\sigma^2(Z_g \mid u_g = 0)$, (2.3.96), into the present context as

$$
\sigma^2(\tilde{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0) = 1 - \frac{\phi[\gamma_2(\boldsymbol{v}_1)]}{Q_2(\boldsymbol{v}_1)} \left\{ \frac{\phi[\gamma_2(\boldsymbol{v}_1)]}{Q_2(\boldsymbol{v}_1)} + \gamma_2(\boldsymbol{v}_1) \right\}, \qquad (5.4.126)
$$

which provides the variance of $\tilde{Z}_2(\boldsymbol{v}_1)$ when an incorrect answer to Item 2 is observed.

Because of (5.4.57), (5.4.66), and (5.4.80), an abbreviated form of (5.4.125) may be written as

$$
\sigma^2(\tilde{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0) = \frac{1}{\sigma^2(Z_2 \mid \boldsymbol{v}_1)} [\mathscr{E}(\dot{Z}_2^2 \mid \boldsymbol{v}_1,u_2 = 0) - \mu^2(\dot{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0)], \qquad (5.4.127)
$$

where the expression in brackets is a variance. This expression, by definition, is the variance of $\dot{Z}_2(\boldsymbol{v}_1)$ when an incorrect answer to Item 2 is observed. The equation for this variance then is

$$
\sigma^2(\dot{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0) = \mathscr{E}(\dot{Z}_2^2 \mid \boldsymbol{v}_1,u_2 = 0) - \mu^2(\dot{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0). \qquad (5.4.128)
$$

A substitution from (5.4.128) into (5.4.127) allows one to write

$$
\sigma^2(\tilde{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0) = \frac{\sigma^2(\dot{Z}_2 \mid \boldsymbol{v}_1,u_2 = 0)}{\sigma^2(Z_2 \mid \boldsymbol{v}_1)}, \qquad (5.4.129)
$$

where $\sigma^2(\dot{Z}_2 \mid \boldsymbol{v}_1,\ u_2 = 0)$ has an identity which after substitution into (5.4.129) provides a more useful expression.

In solving for this identity, it is necessary to examine further the expected value $\mathscr{E}(\dot{Z}_2^2 \mid \boldsymbol{v}_1,u_2 = 0)$. Because of (5.4.66), this expected value of squared mean deviates may be expanded as

$$
\begin{aligned}
\mathscr{E}(\dot{Z}_2^2 \mid \boldsymbol{v}_1,u_2 = 0) &= \int_{-\infty}^{\gamma_2(\boldsymbol{v}_1)} \int_{-\infty}^{\infty} [\dot{\zeta}_2(\boldsymbol{v}_1)]^2\, \phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*(\boldsymbol{v}_1)\, d\tilde{\zeta}_2(\boldsymbol{v}_1) \\
&= \int_{-\infty}^{\gamma_2(\boldsymbol{v}_1)} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid \boldsymbol{v}_1)]^2\, \phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*\, d\tilde{\zeta}_2(\boldsymbol{v}_1) \\
&= \int_{-\infty}^{\gamma_2(\boldsymbol{v}_1)} \int_{-\infty}^{\infty} \zeta_2^2\, \phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*(\boldsymbol{v}_1)\, d\tilde{\zeta}_2(\boldsymbol{v}_1) \\
&\quad - 2\mu(Z_2 \mid \boldsymbol{v}_1) \int_{-\infty}^{\gamma_2(\boldsymbol{v}_1)} \int_{-\infty}^{\infty} \zeta_2\, \phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*(\boldsymbol{v}_1)\, d\tilde{\zeta}_2(\boldsymbol{v}_1) \\
&\quad + \mu^2(Z_2 \mid \boldsymbol{v}_1) \int_{-\infty}^{\gamma_2(\boldsymbol{v}_1)} \int_{-\infty}^{\infty} \phi^{\dagger}[\tilde{\zeta}_2(\boldsymbol{v}_1),\ \tilde{\theta}^*(\boldsymbol{v}_1)]\, d\tilde{\theta}^*(\boldsymbol{v}_1)\, d\tilde{\zeta}_2(\boldsymbol{v}_1),
\end{aligned}
\qquad (5.4.130)
$$

where, as a matter of convenience, the constant, $\mu(Z_2 \mid \nu_1)$ or $\mu^2(Z_2 \mid \nu_1)$, has been moved outside of two of the double integrals. In the last equality in (5.4.130), the first double integral defines $\mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0)$, where one then has

$$\mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0) = \int_{-\infty}^{\gamma_2(\nu_1)} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^\dagger[\tilde{\zeta}_2(\nu_1), \tilde{\theta}^*(\nu_1)] \, d\tilde{\theta}^*(\nu_1) \, d\tilde{\zeta}_2(\nu_1); \qquad (5.4.131)$$

and the second double integral defines $\mu(Z_2 \mid \nu_1, u_2 = 0)$, where one has

$$\mu(Z_2 \mid \nu_1, u_2 = 0) = \int_{-\infty}^{\gamma_2(\nu_1)} \int_{-\infty}^{\infty} \zeta_2 \, \phi^\dagger[\tilde{\zeta}_2(\nu_1), \tilde{\theta}^*(\nu_1)] \, d\tilde{\theta}^*(\nu_1) \, d\tilde{\zeta}_2(\nu_1). \qquad (5.4.132)$$

Substituting from (5.4.78), (5.4.131), and (5.4.132) into (5.4.130), one may write

$$\begin{aligned} &\mathscr{E}(\dot{Z}_2^2 \mid \nu_1, u_2 = 0) \\ &\quad = \mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0) - 2\mu(Z_2 \mid \nu_1) \, \mu(Z_2 \mid \nu_1, u_2 = 0) \\ &\qquad + \frac{\mu^2(Z_2 \mid \nu_1)}{Q_2(\nu_1)} \int_{-\infty}^{\gamma_2(\nu_1)} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(\nu_1), \tilde{\theta}^*(\nu_1)] \, d\tilde{\theta}^*(\nu_1) \, d\tilde{\zeta}_2(\nu_1), \end{aligned} \qquad (5.4.133)$$

where this equation may be simplified to

$$\begin{aligned} &\mathscr{E}(\dot{Z}_2^2 \mid \nu_1, u_2 = 0) \\ &\quad = \mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0) - 2\mu(Z_2 \mid \nu_1) \, \mu(Z_2 \mid \nu_1, u_2 = 0) \\ &\qquad + \mu^2(Z_2 \mid \nu_1) \end{aligned} \qquad (5.4.134)$$

because of (5.4.60). A substitution from (5.4.134) into (5.4.128) now provides

$$\begin{aligned} &\sigma^2(\dot{Z}_2 \mid \nu_1, u_2 = 0) \\ &\quad = \mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0) - 2\mu(Z_2 \mid \nu_1) \, \mu(Z_2 \mid \nu_1, u_2 = 0) \\ &\qquad + \mu^2(Z_2 \mid \nu_1) - \mu^2(\dot{Z}_2 \mid \nu_1, u_2 = 0), \end{aligned} \qquad (5.4.135)$$

where a further substitution from the squared result of (5.4.86) into (5.4.135) yields

$$\begin{aligned} &\sigma^2(\dot{Z}_2 \mid \nu_1, u_2 = 0) \\ &\quad = \mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0) - 2\mu(Z_2 \mid \nu_1) \, \mu(Z_2 \mid \nu_1, u_2 = 0) \\ &\qquad + \mu^2(Z_2 \mid \nu_1) - \mu^2(Z_2 \mid \nu_1, u_2 = 0) \\ &\qquad + 2\mu(Z_2 \mid \nu_1) \, \mu(Z_2 \mid \nu_1, u_2 = 0) - \mu^2(Z_2 \mid \nu_1). \end{aligned} \qquad (5.4.136)$$

The cancellation of terms in (5.4.136) provides

$$\sigma^2(\dot{Z}_2 \mid \nu_1, u_2 = 0) = \mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0) - \mu^2(Z_2 \mid \nu_1, u_2 = 0). \qquad (5.4.137)$$

But, by definition, one has

$$\sigma^2(Z_2 \mid \nu_1, u_2 = 0) = \mathscr{E}(Z_2^2 \mid \nu_1, u_2 = 0) - \mu^2(Z_2 \mid \nu_1, u_2 = 0), \qquad (5.4.138)$$

where a substitution from (5.4.138) into (5.4.137) provides the sought identity

$$\sigma^2(\dot{Z}_2 \mid \nu_1, u_2 = 0) = \sigma^2(Z_2 \mid \nu_1, u_2 = 0). \qquad (5.4.139)$$

The substitution of this identity of $\sigma^2(\dot{Z}_2 \mid \nu_1, u_2 = 0)$ into (5.4.129) now yields

$$\sigma^2(\bar{Z}_2 \mid \nu_1, u_2 = 0) = \frac{\sigma^2(Z_2 \mid \nu_1, u_2 = 0)}{\sigma^2(Z_2 \mid \nu_1)}, \qquad (5.4.140)$$

which because of (5.4.126) may be extended to provide the useful form

$$\sigma^2(\tilde{Z}_2 \mid v_1,u_2 = 0) = \frac{\sigma^2(Z_2 \mid v_1,u_2 = 0)}{\sigma^2(Z_2 \mid v_1)}$$

$$= 1 - \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} \left\{ \frac{\phi[\gamma_2(v_1)]}{Q_2(v_1)} + \gamma_2(v_1) \right\}, \qquad (5.4.141)$$

which allows a substitution into a development from selection theory. This substitution will lead to a convenient expression for the variance of ability given the outcome v_1 when an incorrect answer to free-response Item 2 is observed.

Given the three-parameter normal ogive submodel, the updating assumption, and the previous outcome

The variance of $\tilde{Z}_2(v_1')$ when a correct answer to multiple-choice Item 2 is observed, $\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 1)$, is defined by

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 1) = \mathscr{E}(\tilde{Z}_2^2 \mid v_1',u_2' = 1) - \mu^2(\tilde{Z}_2 \mid v_1',u_2' = 1)$$

$$= \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} [\tilde{\zeta}_2(v_1')]^2 \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} [\tilde{\zeta}_2(v_1')]^2 \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$- \left\{ \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1') \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right.$$

$$\left. + c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1') \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right\}^2, \qquad (5.4.142)$$

where the joint density function $\phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')]$ is provided by (5.4.88). Equations (5.4.88) and (5.4.142) may be viewed as resulting from (3.3.11), (3.3.12), and (3.3.34) after substitutions of $\gamma_2(v_1')$ for γ_g, $\tilde{\zeta}_2(v_1')$ for ζ_g, $\tilde{\theta}^*(v_1')$ for θ, $P_2'(v_1')$ for P_g', and c_2 for c_g. Since $\gamma_2(v_1')$, $\tilde{\zeta}_2(v_1')$, $\tilde{\theta}^*(v_1')$, $P_2'(v_1')$, c_2, and, consequently, $\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 1)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, P_g', c_g, and $\sigma^2(Z_g \mid u_g' = 1)$ in this earlier context, one can then interpret the earlier solution for $\sigma^2(Z_g \mid u_g' = 1)$, (3.3.42), into the present context as

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 1)$$

$$= 1 - \frac{(1 - c_2)\, \phi[\gamma_2(v_1')]}{P_2'(v_1')} \left\{ \frac{(1 - c_2)\, \phi[\gamma_2(v_1')]}{P_2'(v_1')} - \gamma_2(v_1') \right\}, \qquad (5.4.143)$$

which provides the variance of $\tilde{Z}_2(v_1')$ when a correct answer to a multiple-choice Item 2 is observed.

Because of (5.4.57), (5.4.66), and (5.4.90), an abbreviated form of (5.4.142) may be written as

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 1) = \frac{1}{\sigma^2(Z_2 \mid v_1')} [\mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 1) - \mu^2(\dot{Z}_2^2 \mid v_1',u_2' = 1)], \qquad (5.4.144)$$

where the expression in brackets is a variance. This expression, by definition, is the variance of $\dot{Z}_2(v_1')$ when a correct answer to Item 2 is observed. The equation for this variance then is

$$\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 1) = \mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 1) - \mu^2(\dot{Z}_2 \mid v_1',u_2' = 1). \qquad (5.4.145)$$

A substitution from (5.4.145) into (5.4.144) allows one to write

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 1) = \frac{\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 1)}{\sigma^2(Z_2 \mid v_1')}, \qquad (5.4.146)$$

where $\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 1)$ has an identity which after substitution into (5.4.146) provides a more useful expression.

In solving for this identity, it is necessary to examine further the expected value $\mathscr{E}(\dot{Z}_2^2 \mid v_1', u_2' = 1)$. Because of (5.4.66), this expected value of squared mean deviates may be expanded as

$$\mathscr{E}(\dot{Z}_2^2 \mid v_1', u_2' = 1)$$

$$= \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} [\dot{\zeta}_2(v_1')]^2 \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} [\dot{\zeta}_2(v_1')]^2 \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$= \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid v_1')]^2 \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid v_1')]^2 \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$= \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$-2\mu(Z_2 \mid v_1') \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \zeta_2 \, \phi^{*\prime}[\tilde{\zeta}_2(v\tilde{\theta}^*(v''{}_1), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ \mu^2(Z_2 \mid v_1') \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \text{cf1})] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$-2\, c_2\, \mu(Z_2 \mid v_l') \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2 \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ c_2\, \mu^2(Z_2 \mid v_1') \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'), \qquad (5.4.147)$$

where the constant, $\mu(Z_2 \mid v_1')$ or $\mu^2(Z_2 \mid v_1')$, has been moved outside of four of the double integrals for the purpose of convenience. By definition, it is known that

$$\mathscr{E}(Z_2^2 \mid v_1', u_2' = 1)$$

$$= \int_{\gamma_2(v_1')}^{\infty} \int^{\infty} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'), \qquad (5.4.148)$$

which may be substituted into (5.4.147). After this substitution and some rearrangement, one may write

$$\mathscr{E}(\dot{Z}_2^2 \mid v_1', u_2' = 1) = \mathscr{E}(Z_2^2 \mid v_1', u_2' = 1)$$

$$- 2\mu(Z_2 \mid v_1') \left\{ \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \zeta_2 \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right.$$

$$\left. + c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2 \, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right\}$$

$$+ \mu^2(Z_2 \mid v_1') \left\{ \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right.$$

$$\left. + c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1') \right\}, \qquad (5.4.149)$$

an equation that can be further simplified. By definition, one has

$$
\begin{aligned}
&\mu(Z_2 \mid v_1', u_2' = 1) \\
&\quad = \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \zeta_2\, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1') \\
&\qquad + c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2\, \phi^{*\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1'),
\end{aligned}
\tag{5.4.150}
$$

which may now be substituted into (5.4.149). After substitutions from (5.4.88) and (5.4.150) into (5.4.149), it is known that

$$
\begin{aligned}
&\mathscr{E}(\dot{Z}_2^2 \mid v_1', u_2' = 1) \\
&\quad = \mathscr{E}(Z^2 \mid v_1', u_2' = 1) - 2\mu(Z_2 \mid v_1')\, \mu(Z_2 \mid v_1', u_2' = 1) \\
&\qquad + \frac{\mu^2(Z_2 \mid v_1')}{P_2'(v_1')} \left\{ \int_{\gamma_2(v_1')}^{\infty} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1') \right. \\
&\qquad \left. + c_2 \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')]\, d\tilde{\theta}^*(v_1')\, d\tilde{\zeta}_2(v_1') \right\},
\end{aligned}
\tag{5.4.151}
$$

which can be written as

$$
\begin{aligned}
&\mathscr{E}(\dot{Z}_2^2 \mid v_1', u_2' = 1) \\
&\quad = \mathscr{E}(Z_2^2 \mid v_1', u_2' = 1) - 2\mu(Z_2 \mid v_1')\, \mu(Z_2 \mid v_1', u_2' = 1) \\
&\qquad + \mu^2(Z_2 \mid v_1')
\end{aligned}
\tag{5.4.152}
$$

because of (5.4.62). A substitution from (5.4.152) into (5.4.145) now leads to

$$
\begin{aligned}
&\sigma^2(\dot{Z}_2 \mid v_1', u_2' = 1) \\
&\quad = \mathscr{E}(Z_2^2 \mid v_1', u_2' = 1) - 2\mu(Z_2 \mid v_1')\, \mu(Z_2 \mid v_1', u_2' = 1) \\
&\qquad + \mu^2(Z_2 \mid v_1') - \mu^2(\dot{Z}_2 \mid v_1', u_2' = 1),
\end{aligned}
\tag{5.4.153}
$$

where a further substitution from the squared result of (5.4.96) yields

$$
\begin{aligned}
\sigma^2(\dot{Z}_2 \mid v_1', u_2' = 1) &= \mathscr{E}(Z_2^2 \mid v_1', u_2' = 1) - 2\mu(Z_2 \mid v_1')\, \mu(Z_2 \mid v_1', u_2' = 1) \\
&\quad + \mu^2(Z_2 \mid v_1') - \mu^2(Z_2 \mid v_1', u_2' = 1) \\
&\quad + 2\mu(Z_2 \mid v_1')\, \mu(Z_2 \mid v_1', u_2' = 1) - \mu^2(Z_2 \mid v_1').
\end{aligned}
\tag{5.4.154}
$$

The cancellation of terms in (5.4.154) provides

$$
\sigma^2(\dot{Z}_2 \mid v_1', u_2' = 1) = \mathscr{E}(Z_2^2 \mid v_1', u_2' = 1) - \mu^2(Z_2 \mid v_1', u_2' = 1). \tag{5.4.155}
$$

But, by definition, one has

$$
\sigma^2(Z_2 \mid v_1', u_2' = 1) = \mathscr{E}(Z_2^2 \mid v_1', u_2' = 1) - \mu^2(Z_2 \mid v_1', u_2' = 1), \tag{5.4.156}
$$

where a substitution from (5.4.156) into (5.4.155) provides the sought identity

$$
\sigma^2(\dot{Z}_2 \mid v_1', u_2' = 1) = \sigma^2(Z_2 \mid v_1', u_2' = 1). \tag{5.4.157}
$$

The substitution of this identity of $\sigma^2(\dot{Z}_2 \mid v_1', u_2' = 1)$ into (5.4.146) yields

$$
\sigma^2(\tilde{Z}_2 \mid v_1', u_2' = 1) = \frac{\sigma^2(Z_2 \mid v_1', u_2' = 1)}{\sigma^2(Z_2 \mid v_1')}, \tag{5.4.158}
$$

which because of (5.4.143) may be extended to provide the useful form

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 1) = \frac{\sigma^2(Z_2 \mid v_1',u_2' = 1)}{\sigma^2(Z_2 \mid v_1')}$$

$$= 1 - \frac{(1 - c_2)\,\phi[\gamma_2(v_1')]}{P_2'(v_1')}\left\{\frac{(1 - c_2)\,\phi[\gamma_2(v_1')]}{P_2'(v_1')} - \gamma_2(v_1')\right\}, \qquad (5.4.159)$$

which allows a substitution into a development from selection theory. This substitution will lead to a convenient expression for the variance of ability given the outcome v_1' when a correct answer to multiple-choice Item 2 is observed.

The variance of $\tilde{Z}_2(v_1')$ when an incorrect answer to multiple-choice Item 2 is observed, $\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0)$, is by definition

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0) = \mathscr{E}(\tilde{Z}_2^2 \mid v_1',u_2' = 0) - \mu^2(\tilde{Z}_2 \mid v_1',u_2' = 0)$$

$$= (1 - c_2)\int_{-\infty}^{\gamma_2(v_1')}\int_{-\infty}^{\infty} [\tilde{\zeta}_2(v_1')]^2\,\phi^{\dagger'}[\tilde{\zeta}_2(v_1'),\,\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')$$

$$-\left\{(1 - c_2)\int_{-\infty}^{\gamma_2(v_1')}\int_{-\infty}^{\infty} \tilde{\zeta}_2(v_1')\,\phi^{\dagger'}[\tilde{\zeta}_2(v_1'),\,\tilde{\theta}^*(v_1')]\,d\tilde{\theta}^*(v_1')\,d\tilde{\zeta}_2(v_1')\right\}^2,$$

$$(5.4.160)$$

where the joint density function $\phi^{\dagger'}[\tilde{\zeta}_2(v_1'),\,\tilde{\theta}^*(v_1')]$ is provided by (5.4.98). Equations (5.4.98) and (5.4.160) may be viewed as resulting from (3.3.22), (3.3.23), and (3.3.45) after substitutions of $\gamma_2(v_1')$ for γ_g, $\tilde{\zeta}_2(v_1')$ for ζ_g, $\tilde{\theta}^*(v_1')$ for θ, $Q_2'(v_1')$ for Q_g', and c_2 for c_g. Since $\gamma_2(v_1')$, $\tilde{\zeta}_2(v_1')$, $\tilde{\theta}^*(v_1')$, $Q_2'(v_1')$, $Q_2(v_1')$, c_2, and, consequently, $\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0)$ have respective definitions that correspond to those of γ_g, ζ_g, θ, Q_g', Q_g, c_g, and $\sigma^2(Z_g \mid u_g' = 0)$ in this earlier context, one can then interpret the earlier solution for $\sigma^2(Z_g \mid u_g' = 0)$, (3.3.50), into the present context as

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0) = 1 - \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')}\left\{\frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} + \gamma_2(v_1')\right\}, \qquad (5.4.161)$$

which provides the variance of $\tilde{Z}_2(v_1')$ when an incorrect answer to Item 2 is observed.

Because of (5.4.57), (5.4.66), and (5.4.100), an abbreviated form of (5.4.160) may be written as

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0) = \frac{1}{\sigma^2(Z_2 \mid v_1')}\,[\mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 0) - \mu^2(\dot{Z}_2 \mid v_1',u_2' = 0)], \qquad (5.4.162)$$

where the expression in brackets is a variance. This expression, by definition, is the variance of $\dot{Z}_2(v_1')$ when an incorrect answer to Item 2 is observed. The equation for this variance then is

$$\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0) = \mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 0) - \mu^2(\dot{Z}_2 \mid v_1',u_2' = 0). \qquad (5.4.163)$$

A substitution from (5.4.163) into (5.4.162) allows one to write

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0) = \frac{\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0)}{\sigma^2(Z_2 \mid v_1')}, \qquad (5.4.164)$$

where $\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0)$ has an identity which after substitution into (5.4.164) provides a more useful expression.

In solving for this identity, it is necessary to examine further the expected value $\mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 0)$. Because of (5.4.66), this expected value of squared mean deviates may be expanded as

$$\mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 0) = (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} [\dot{\zeta}_2(v_1')]^2 \, \phi^{\dagger\prime}[\dot{\zeta}_2(v_1'), \dot{\theta}^*(v_1')] \, d\dot{\theta}^*(v_1') \, d\dot{\zeta}_2(v_1')$$

$$= (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} [\zeta_2 - \mu(Z_2 \mid v_1')]^2 \, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$= (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$- 2(1 - c_2) \, \mu(Z_2 \mid v_1') \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2 \, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1')$$

$$+ (1 - c_2) \, \mu^2(Z_2 \mid v_1') \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'), \qquad (5.4.165)$$

where, as a matter of convenience, the constant, $\mu(Z_2 \mid v_1')$ or $\mu^2(Z_2 \mid v_1')$, has been moved outside of two of the double integrals. In the last equality in (5.4.165), the product of $(1 - c_2)$ and the first double integral defines $\mathscr{E}(Z_2^2 \mid v_1',u_2' = 0)$ where one then has

$$\mathscr{E}(Z_2^2 \mid v_1',u_2' = 0)$$
$$= (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2^2 \, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'), \qquad (5.4.166)$$

and the product of $(1 - c_2)$ and the second double integral defines $\mu(Z_2 \mid v_1',u_2' = 0)$, where one has

$$\mu(Z_2 \mid v_1',u_2' = 0)$$
$$= (1 - c_2) \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \zeta_2 \, \phi^{\dagger\prime}[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'). \qquad (5.4.167)$$

Substituting from (5.4.98), (5.4.166), and (5.4.167) into (5.4.165), one may write

$$\mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 0)$$
$$= \mathscr{E}(Z_2^2 \mid v_1',u_2' = 0) - 2\mu(Z_2 \mid v_1') \, \mu(Z_2 \mid v_1',u_2' = 0)$$
$$+ (1 - c_2) \frac{\mu^2(Z_2 \mid v_1')}{Q_2'(v_1')} \int_{-\infty}^{\gamma_2(v_1')} \int_{-\infty}^{\infty} \phi[\tilde{\zeta}_2(v_1'), \tilde{\theta}^*(v_1')] \, d\tilde{\theta}^*(v_1') \, d\tilde{\zeta}_2(v_1'), \qquad (5.4.168)$$

where this equation may be simplified to

$$\mathscr{E}(\dot{Z}_2^2 \mid v_1',u_2' = 0) = \mathscr{E}(Z_2^2 \mid v_1',u_2' = 0) - 2\mu(Z_2 \mid v_1') \, \mu(Z_2 \mid v_1',u_2' = 0)$$
$$+ \mu^2(Z_2 \mid v_1'), \qquad (5.4.169)$$

because of (5.4.64). A substitution from (5.4.169) into (5.4.163) now provides

$$\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0)$$
$$= \mathscr{E}(Z_2^2 \mid v_1',u_2' = 0) - 2\mu(Z_2 \mid v_1') \, \mu(Z_2 \mid v_1',u_2' = 0)$$
$$+ \mu^2(Z_2 \mid v_1') - \mu^2(\dot{Z}_2 \mid v_1',u_2' = 0), \qquad (5.4.170)$$

where a further substitution from the squared result of (5.4.106) into (5.4.170) yields

$$\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0) = \mathscr{E}(Z_2^2 \mid v_1',u_2' = 0) - 2\mu(Z_2 \mid v_1') \, \mu(Z_2 \mid v_1',u_2' = 0)$$
$$+ \mu^2(Z_2 \mid v_1') - \mu^2(Z_2 \mid v_1',u_2' = 0)$$
$$+ 2\mu(Z_2 \mid v_1') \, \mu(Z_2 \mid v_1',u_2' = 0) - \mu^2(Z_2 \mid v_1'). \qquad (5.4.171)$$

The cancellation of terms in (5.4.171) provides

$$\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0) = \mathscr{E}(Z_2^2 \mid v_1',u_2' = 0) - \mu^2(Z_2 \mid v_1',u_2' = 0). \qquad (5.4.172)$$

But, by definition, one has

$$\sigma^2(Z_2 \mid v_1',u_2' = 0) = \mathscr{E}(Z_2^2 \mid v_1',u_2' = 0) - \mu^2(Z_2 \mid v_1',u_2' = 0), \tag{5.4.173}$$

where a substitution from (5.4.173) into (5.4.172) provides the sought identity

$$\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0) = \sigma^2(Z_2 \mid v_1',u_2' = 0). \tag{5.4.174}$$

The substitution of this identity of $\sigma^2(\dot{Z}_2 \mid v_1',u_2' = 0)$ into (5.4.164) yields

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0) = \frac{\sigma^2(Z_2 \mid v_1',u_2' = 0)}{\sigma^2(Z_2 \mid v_1')}. \tag{5.4.175}$$

Because of (5.4.161), equation (5.4.175) may now be extended to provide the useful form

$$\sigma^2(\tilde{Z}_2 \mid v_1',u_2' = 0) = \frac{\sigma^2(Z_2 \mid v_1',u_2' = 0)}{\sigma^2(Z_2 \mid v_1')}$$

$$= 1 - \frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')}\left\{\frac{\phi[\gamma_2(v_1')]}{Q_2(v_1')} + \gamma_2(v_1')\right\}, \tag{5.4.176}$$

which allows a substitution into a development from selection theory. This substitution will lead to a convenient expression for the variance of ability given the outcome v_1', when an incorrect answer to multiple-choice Item 2 is observed.

The derivations culminating in (5.4.122), (5.4.139), (5.4.157), and (5.4.174) demonstrate the principle that the variance of a variable is unchanged by the addition of a constant. The continuous variable $\dot{Z}_2(v_1^*)$ equals the continuous variable Z_2 plus a constant. In the general case, this additive constant is the negative of $\mu(Z_2 \mid v_1^*)$.

A generalized form for the variance of the standardized continuous variable $\tilde{Z}_2(v_1^*)$ given the realizations of the binary variable U_2^* may now be written. As can be deduced from (5.4.123), (5.4.140), (5.4.158), and (5.4.175), this expression takes the form

$$\sigma^2(\tilde{Z}_2 \mid v_1^*,U_2^*) = \frac{\sigma^2(Z_2 \mid v_1^*,U_2^*)}{\sigma^2(Z_2 \mid v_1^*)}. \tag{5.4.177}$$

The Correlation Between Z_3 and Θ^* Given the Outcome Vector $\mathbf{v}_2^*$, $\rho(Z_3,\Theta \mid \mathbf{v}_2^*)$

In obtaining an explicit solution for $\rho(Z_3,\Theta \mid v_2^*)$ one begins by deriving expressions for $\rho^2(Z_2,Z_3 \mid v_1^*)$ and $\rho^2(Z_2,\Theta \mid v_1^*)$ for later substitution into the first two equalities in (5.3.56). In solving for $\rho^2(Z_2,Z_3 \mid v_1^*)$, one obtains an explicit solution for $\sigma(Z_3 \mid v_2^*)$ from the first two equalities in (5.3.52). This solution yields

$$\sigma(Z_3 \mid v_2^*) = \frac{\rho(Z_2,Z_3 \mid v_1^*)}{\rho(Z_2,Z_3 \mid v_2^*)}\,\frac{\sigma(Z_3 \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)}\,\sigma(Z_2 \mid v_2^*) \tag{5.4.178}$$

where, upon squaring (5.4.178) and substituting from the result into the first two equalities in (5.3.54), some rearrangement provides

$$\sigma^2(Z_3|v_1^*)\,[1 - \rho^2(Z_2,Z_3|v_1^*)] = \rho^2(Z_2,Z_3|v_1^*)\,\frac{\sigma^2(Z_3|v_1^*)\,\sigma^2(Z_2|v_2^*)}{\sigma^2(Z_2|v_1^*)}\left[\frac{1}{\rho^2(Z_2,Z_3|v_2^*)} - 1\right]. \tag{5.4.179}$$

Dividing (5.4.179) by the product $\sigma^2(Z_3 \mid v_1^*)\,\rho^2(Z_2,Z_3 \mid v_1^*)$, one has

$$\frac{1}{\rho^2(Z_2,Z_3 \mid v_1^*)} - 1 = \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)}\left[\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1\right], \tag{5.4.180}$$

where an explicit solution for $\rho^2(Z_2,Z_3 \mid v_1^*)$ yields

$$\rho^2(Z_2,Z_3 \mid v_1^*) = \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)}\left[\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1\right]\right\}^{-1}. \tag{5.4.181}$$

In solving for $\rho^2(Z_2,\Theta \mid v_1^*)$, one obtains an explicit solution for $\sigma(\Theta^* \mid v_2^*)$ from the first two equalities in (5.3.53). This solution yields

$$\sigma(\Theta^* \mid v_2^*) = \frac{\rho(Z_2,\Theta \mid v_1^*)}{\rho(Z_2,\Theta \mid v_2^*)} \frac{\sigma(\Theta^* \mid v_1^*)}{\sigma(Z_2 \mid v_1^*)} \sigma(Z_2 \mid v_2^*) \tag{5.4.182}$$

where, upon squaring (5.4.182) and substituting from the result into the first two equalities in (5.3.55), some rearrangement provides

$$\sigma^2(\Theta^*|v_1^*)\,[1 - \rho^2(Z_2,\Theta|v_1^*)] = \rho^2(Z_2,\Theta|v_1^*) \frac{\sigma^2(\Theta^*|v_1^*)\,\sigma^2(Z_2|v_2^*)}{\sigma^2(Z_2|v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta|v_2^*)} - 1\right]. \tag{5.4.183}$$

Dividing (5.4.183) by the product $\sigma^2(\Theta^* \mid v_1^*)\,\rho^2(Z_2,\Theta \mid v_1^*)$, one has

$$\frac{1}{\rho^2(Z_2,\Theta \mid v_1^*)} - 1 = \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1\right], \tag{5.4.184}$$

where an explicit solution for $\rho^2(Z_2,\Theta \mid v_1^*)$ now yields

$$\rho^2(Z_2,\Theta \mid v_1^*) = \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1\right]\right\}^{-1}. \tag{5.4.185}$$

After appropriately substituting the results of (5.4.181) and (5.4.185) into the first two equalities in (5.3.56), one obtains

$$\frac{\rho(Z_3,\Theta \mid v_2^*) - \rho(Z_2,Z_3 \mid v_2^*)\,\rho(Z_2,\Theta \mid v_2^*)}{\sqrt{1 - \rho^2(Z_2,Z_3 \mid v_2^*)}\,\sqrt{1 - \rho^2(Z_2,\Theta \mid v_2^*)}}$$

$$= \frac{\rho(Z_3,\Theta \mid v_1^*) - \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2, Z_3 \mid v_2^*)} - 1\right]\right\}^{-.5} \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2, \Theta \mid v_2^*)} - 1\right]\right\}^{-.5}}{\sqrt{1 - \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1\right]\right\}^{-1}}\,\sqrt{1 - \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1\right]\right\}^{-1}}}, \tag{5.4.186}$$

where the equality on the right side can be simplified. By placing the terms under the radicals on common denominators and subtracting terms in the numerators, one obtains

$$\frac{\rho(Z_3,\Theta \mid v_1^*) - \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1\right]\right\}^{-.5} \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1\right]\right\}^{-.5}}{\sqrt{\dfrac{\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1\right]}{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1\right]}}\,\sqrt{\dfrac{\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1\right]}{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1\right]}}}. \tag{5.4.187}$$

Multiplication of the numerator and denominator of (5.4.187) by the square roots of the denominators under the radicals, along with some later rearrangement, leads to

$$\frac{\rho(Z_3,\Theta \mid v_1^*) \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1\right]\right\}^{.5} \left\{1 + \frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \left[\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1\right]\right\}^{.5} - 1}{\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} \sqrt{\frac{1}{\rho^2(Z_2,Z_3 \mid v_2^*)} - 1}\,\sqrt{\frac{1}{\rho^2(Z_2,\Theta \mid v_2^*)} - 1}}, \tag{5.4.188}$$

where multiplication of the numerator and denominator by the ratio $\dfrac{\sigma^2(Z_2 \mid v_1^*)}{\sigma^2(Z_2 \mid v_2^*)}$, along with some later rearrangement, leads to

$$\frac{\rho(Z_3,\Theta \mid \nu_1^*) \left\{ \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} + \dfrac{1}{\rho^2(Z_2,Z_3 \mid \nu_2^*)} - 1 \right\}^{.5} \left\{ \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} + \dfrac{1}{\rho^2(Z_2,\Theta \mid \nu_2^*)} - 1 \right\}^{.5} - \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)}}{\dfrac{1}{\rho(Z_2,Z_3 \mid \nu_2^*)} \sqrt{1 - \rho^2(Z_2,Z_3 \mid \nu_2^*)} \; \dfrac{1}{\rho(Z_2,\Theta \mid \nu_2^*)} \sqrt{1 - \rho^2(Z_2,\Theta \mid \nu_2^*)}} .$$

(5.4.189)

Multiplication of the numerator and denominator in (5.4.189) by $\rho(Z_2,Z_3 \mid \nu_2^*)$ and $\rho(Z_2,\Theta \mid \nu_2^*)$, along with subsequent rearrangement, leads to

$$\frac{\rho(Z_3,\Theta \mid \nu_1^*) \left\{ 1 - \rho^2(Z_2,Z_3 \mid \nu_2^*) \left[1 - \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} \left\{ 1 - \rho^2(Z_2,\Theta \mid \nu_2^*) \left[1 - \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} - \rho(Z_2,Z_3 \mid \nu_2^*)\, \rho(Z_2,\Theta \mid \nu_2^*) \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)}}{\sqrt{1 - \rho^2(Z_2,Z_3 \mid \nu_2^*)} \sqrt{1 - \rho^2(Z_2,\Theta \mid \nu_2^*)}}$$

(5.4.190)

which may be substituted back into (5.4.186) for the expression on the right side of the equality. This substitution provides

$$\frac{\rho(Z_3,\Theta \mid \nu_2^*) - \rho(Z_2,Z_3 \mid \nu_2^*)\, \rho(Z_2,\Theta \mid \nu_2^*)}{\sqrt{1 - \rho^2(Z_2,Z_3 \mid \nu_2^*)} \sqrt{1 - \rho^2(Z_2,\Theta \mid \nu_2^*)}}$$

$$= \frac{\rho(Z_3,\Theta \mid \nu_1^*) \left\{ 1 - \rho^2(Z_2,Z_3 \mid \nu_2^*) \left[1 - \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} \left\{ 1 - \rho^2(Z_2,\Theta \mid \nu_2^*) \left[1 - \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} - \rho(Z_2,Z_3 \mid \nu_2^*)\, \rho(Z_2,\Theta \mid \nu_2^*) \dfrac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)}}{\sqrt{1 - \rho^2(Z_2,Z_3 \mid \nu_2^*)} \sqrt{1 - \rho^2(Z_2,\Theta \mid \nu_2^*)}} ,$$

(5.4.191)

where multiplication of both sides of this equation by the common denominator yields

$$\begin{aligned} &\rho(Z_3,\Theta \mid \nu_2^*) - \rho(Z_2,Z_3 \mid \nu_2^*)\, \rho(Z_2,\Theta \mid \nu_2^*) \\ &\quad = \rho(Z_3,\Theta \mid \nu_1^*) \left\{ 1 - \rho^2(Z_2,Z_3 \mid \nu_2^*) \left[1 - \frac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} \left\{ 1 - \rho^2(Z_2,\Theta \mid \nu_2^*) \left[1 - \frac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} \\ &\qquad - \rho(Z_2,Z_3 \mid \nu_2^*)\, \rho(Z_2,\Theta \mid \nu_2^*) \frac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} . \end{aligned} \qquad (5.4.192)$$

Through transposing and combining expressions in (5.4.192) one obtains

$$\begin{aligned} &\rho(Z_3,\Theta \mid \nu_2^*) \\ &\quad = \rho(Z_3,\Theta \mid \nu_1^*) \left\{ 1 - \rho^2(Z_2,Z_3 \mid \nu_2^*) \left[1 - \frac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} \left\{ 1 - \rho^2(Z_2,\Theta \mid \nu_2^*) \left[1 - \frac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right] \right\}^{.5} \\ &\qquad + \rho(Z_2,Z_3 \mid \nu_2^*)\, \rho(Z_2,\Theta \mid \nu_2^*) \left[1 - \frac{\sigma^2(Z_2 \mid \nu_1^*)}{\sigma^2(Z_2 \mid \nu_2^*)} \right], \end{aligned} \qquad (5.4.193)$$

which allows one to proceed to a convenient expression for an explicit solution to $\rho(Z_3,\Theta \mid \nu_2^*)$.

Solving the first two equalities in both (5.3.52) and (5.3.53) explicitly for $\rho(Z_2,Z_3 \mid \nu_2^*)$ and $\rho(Z_2,\Theta \mid \nu_2^*)$, respectively, one obtains

$$\rho(Z_2,Z_3 \mid \nu_2^*) = \rho(Z_2,Z_3 \mid \nu_1^*) \frac{\sigma(Z_3 \mid \nu_1^*)\, \sigma(Z_2 \mid \nu_2^*)}{\sigma(Z_2 \mid \nu_1^*)\, \sigma(Z_3 \mid \nu_2^*)} \qquad (5.4.194)$$

and

$$\rho(Z_2,\Theta \mid \nu_2^*) = \rho(Z_2,\Theta \mid \nu_1^*) \frac{\sigma(\Theta^* \mid \nu_1^*)\, \sigma(Z_2 \mid \nu_2^*)}{\sigma(Z_2 \mid \nu_1^*)\, \sigma(\Theta^* \mid \nu_2^*)} , \qquad (5.4.195)$$

where squared and unsquared substitutions from (5.4.194) and (5.4.195) into (5.4.193), along with later rearrangement, lead to

$$\rho(Z_3,\Theta \mid v_2^*)$$

$$= \rho(Z_3,\Theta \mid v_1^*) \sqrt{\left\{1 - \rho^2(Z_2,Z_3 \mid v_1^*) \frac{\sigma^2(Z_3 \mid v_1^*)}{\sigma^2(Z_3 \mid v_2^*)} \left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]\right\} \left\{1 - \rho^2(Z_2,\Theta \mid v_1^*) \frac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^* \mid v_2^*)} \left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]\right\}}$$

$$+ \rho(Z_2,Z_3 \mid v_1^*)\, \rho(Z_2,\Theta \mid v_1^*) \frac{\sigma(Z_3 \mid v_1^*)\, \sigma(\Theta^* \mid v_1^*)}{\sigma(Z_3 \mid v_2^*)\, \sigma(\Theta^* \mid v_2^*)} \left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]. \tag{5.4.196}$$

In simplifying (5.4.196) further, one will want to substitute for $\rho(Z_2,Z_3 \mid v_1^*)$. A convenient expression for this correlation will now be derived for later substitution into (5.4.196). In the restricted updating procedure, $\Theta^*(v_1^*)$ is assumed to be normally distributed. Thus (1.1.12) may be rewritten in this context as

$$\boldsymbol{Z}(v_1^*) = \boldsymbol{\beta}(v_1^*)\, \Theta^*(v_1^*) + \boldsymbol{\Delta} \text{ for all } Z_g(v_1^*) \text{ where } g \neq g^{(1)}. \tag{5.4.197}$$

Since linear combinations of normally distributed variables are also normally distributed (Anderson, 1958), one has

$$\boldsymbol{Z}(v_1^*) \sim N\{\boldsymbol{\beta}(v_1^*)\, \mu(\Theta^* \mid v_1^*),\, \Sigma[Z(v_1^*),\, Z(v_1^*)]\}, \tag{5.4.198}$$

where the $(p-1)$-by-$(p-1)$ variance-covariance matrix $\Sigma[Z(v_1^*),\, Z(v_1^*)]$ can be decomposed into

$$\Sigma[Z(v_1^*),\, Z(v_1^*)] = \boldsymbol{\beta}(v_1^*)\, \boldsymbol{\beta}(v_1^*)'\, \sigma^2(\Theta^* \mid v_1^*) + \Sigma(\Delta,\Delta). \tag{5.4.199}$$

Because the variance-covariance matrix $\Sigma(\Delta,\Delta)$ is diagonal, that is measurement errors are undisturbed by prior *explicit selection* on Z_1 and are independent from Z_g to Z_h where g does not equal h for the remaining items, one has as the off-diagonal element in the gth row an hth column of the matrix $\Sigma[Z(v_1^*),\, Z(v_1^*)]$, the covariance

$$\sigma[Z_g(v_1^*),\, Z_h(v_1^*)] = \beta_g(v_1^*)\, \beta_h(v_1^*)\, \sigma^2(\Theta^* \mid v_1^*). \tag{5.4.200}$$

In (5.4.200), the regression coefficients $\beta_g(v_1^*)$ and $\beta_h(v_1^*)$, those which predict $Z_g(v_1^*)$ and $Z_h(v_1^*)$ given $\Theta^*(v_1^*)$, are by definition

$$\beta_g(v_1^*) = \rho(Z_g,\Theta \mid v_1^*) \frac{\sigma(Z_g \mid v_1^*)}{\sigma(\Theta^* \mid v_1^*)} \tag{5.4.201}$$

and

$$\beta_h(v_1^*) = \rho(Z_h,\Theta \mid v_1^*) \frac{\sigma(Z_h \mid v_1^*)}{\sigma(\Theta^* \mid v_1^*)}, \tag{5.4.202}$$

respectively. After substituting from (5.4.201) and (5.4.202) into (5.4.200), it is known that

$$\sigma[Z_g(v_1^*),\, Z_h(v_1^*)] = \rho(Z_g,\Theta \mid v_1^*)\, \rho(Z_h,\Theta \mid v_1^*)\, \sigma(Z_g \mid v_1^*)\, \sigma(Z_h \mid v_1^*) \tag{5.4.203}$$

where division of (5.4.203) by the product $\sigma(Z_g \mid v_1^*)\, \sigma(Z_h \mid v_1^*)$ yields

$$\frac{\sigma[Z_g(v_1^*),\, Z_h(v_1^*)]}{\sigma(Z_g \mid v_1^*)\, \sigma(Z_h \mid v_1^*)} = \rho(Z_g,\Theta \mid v_1^*)\, \rho(Z_h,\Theta \mid v_1^*). \tag{5.4.204}$$

One then finds that the equality on the left in (5.4.204) is $\rho(Z_g,Z_h \mid v_1^*)$, by definition. One therefore has

$$\rho(Z_g,Z_h \mid v_1^*) = \rho(Z_g,\Theta \mid v_1^*)\, \rho(Z_h,\Theta \mid v_1^*) \tag{5.4.205}$$

for substitution into (5.4.196).

After squared and unsquared substitution from (5.4.205) into (5.4.196), one obtains

$$\rho(Z_3,\Theta \mid v_2^*)$$

$$= \rho(Z_3,\Theta \mid v_1^*) \sqrt{\left\{1 - \rho^2(Z_2,\Theta \mid v_1^*)\, \rho^2(Z_3,\Theta \mid v_1^*) \frac{\sigma^2(Z_3 \mid v_1^*)}{\sigma^2(Z_3 \mid v_2^*)} \left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]\right\} \left\{1 - \rho^2(Z_2,\Theta \mid v_1^*) \frac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^* \mid v_2^*)} \left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma(Z_2 \mid v_1^*)} - 1\right]\right\}}$$

$$+ \rho^2(Z_2,\Theta \mid v_1^*)\, \rho(Z_3,\Theta \mid v_1^*) \frac{\sigma(Z_3 \mid v_1^*)\, \sigma(\Theta^* \mid v_1^*)}{\sigma(Z_3 \mid v_2^*)\, \sigma(\Theta^* \mid v_2^*)} \left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right] \tag{5.4.206}$$

where one will want to substitute for $\sigma^2(Z_3 \mid v_2^*)$ and $\sigma(Z_3 \mid v_2^*)$. In solving the first two equalities in (5.3.54) explicitly for $\sigma^2(Z_3 \mid v_2^*)$, one proceeds by transposition to obtain

$$\sigma^2(Z_3 \mid v_2^*) = \sigma^2(Z_3 \mid v_1^*)\,[1 - \rho^2(Z_2,Z_3 \mid v_1^*)] + \rho^2(Z_2,Z_3 \mid v_2^*)\,\sigma^2(Z_3 \mid v_2^*), \tag{5.4.207}$$

where squared substitution from (5.4.194) into (5.4.207) leads to

$$\sigma^2(Z_3 \mid v_2^*) = \sigma^2(Z_3 \mid v_1^*)\left\{1 + \rho^2(Z_2,Z_3 \mid v_1^*)\left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]\right\}, \tag{5.4.208}$$

which may be rewritten as

$$\sigma^2(Z_3 \mid v_2^*) = \sigma^2(Z_3 \mid v_1^*)\left\{1 + \rho^2(Z_2,\Theta \mid v_1^*)\,\rho^2(Z_3,\Theta \mid v_1^*)\left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]\right\} \tag{5.4.209}$$

because of (5.4.205). Squared and unsquared substitution from (5.4.209) into (5.4.206) now allows one to write

$$\begin{aligned}\rho(Z_3,\Theta \mid v_2^*)\\ = \rho(Z_3,\Theta \mid v_1^*)&\sqrt{\left\{1 - \frac{\rho^2(Z_2,\Theta \mid v_1^*)\,\rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}{1 + \rho^2(Z_2,\Theta \mid v_1^*)\,\rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}\right\}\left\{1 - \rho^2(Z_2,\Theta \mid v_1^*)\frac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^* \mid v_2^*)}\left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]\right\}}\\ &+ \frac{\rho^2(Z_2,\Theta \mid v_1^*)\,\rho(Z_3,\Theta \mid v_1^*)\,\dfrac{\sigma(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_2^*)}\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}{\sqrt{1 + \rho^2(Z_2,\Theta \mid v_1^*)\,\rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}},\end{aligned} \tag{5.4.210}$$

where placing the terms in the first braces under the first radical to the right of the equal sign over a common denominator, subtracting in the numerator and multiplying the resulting expression by the other expression in braces under the same radical yields

$$\begin{aligned}\rho(Z_3,\Theta \mid v_2^*) = \rho(Z_3,\Theta \mid v_1^*)&\sqrt{\frac{1 - \rho^2(Z_2,\Theta \mid v_1^*)\,\dfrac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^* \mid v_2^*)}\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}{1 + \rho^2(Z_2,\Theta \mid v_1^*)\,\rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}}\\ &+ \frac{\rho^2(Z_2,\Theta \mid v_1^*)\,\rho(Z_3,\Theta \mid v_1^*)\,\dfrac{\sigma(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_2^*)}\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}{\sqrt{1 + \rho^2(Z_2,\Theta \mid v_1^*)\,\rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]}}.\end{aligned} \tag{5.4.211}$$

In (5.4.211) there are four occurrences of the same bracketed expression. As a result, one will want to solve explicitly for this expression in order to provide further simplification in (5.4.211) through substitution. One begins by rearranging the first two equalities in (5.3.55) to obtain

$$\sigma^2(\Theta^* \mid v_2^*) = \sigma^2(\Theta^* \mid v_1^*)\,[1 - \rho^2(Z_2,\Theta \mid v_1^*)] + \rho^2(Z_2,\Theta \mid v_2^*)\,\sigma^2(\Theta^* \mid v_2^*)\,, \tag{5.4.212}$$

where squared substitution from (5.4.195) into (5.4.212) allows one to write

$$\sigma^2(\Theta^* \mid v_2^*) = \sigma^2(\Theta^* \mid v_1^*)\left\{1 + \rho^2(Z_2,\Theta \mid v_1^*)\left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right]\right\}. \tag{5.4.213}$$

Through division of (5.4.213) by $\sigma^2(\Theta^* \mid v_1^*)$, one obtains

$$\frac{\sigma^2(\Theta^* \mid v_2^*)}{\sigma^2(\Theta^* \mid v_1^*)} = 1 + \rho^2(Z_2,\Theta \mid v_1^*)\left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right], \tag{5.4.214}$$

where an explicit solution for the bracketed expression yields

$$\left[\frac{\sigma^2(Z_2 \mid v_2^*)}{\sigma^2(Z_2 \mid v_1^*)} - 1\right] = \frac{1}{\rho^2(Z_2,\Theta \mid v_1^*)}\left[\frac{\sigma^2(\Theta^* \mid v_2^*)}{\sigma^2(\Theta^* \mid v_1^*)} - 1\right], \tag{5.4.215}$$

the sought solution. After the four substitutions from (5.4.215) into (5.4.211), the cancellation of several terms and subsequent multiplications lead to

$$\rho(Z_3,\Theta \mid v_2^*) = \rho(Z_3,\Theta \mid v_1^*)\sqrt{\frac{1 - \left[1 - \dfrac{\sigma^2(\Theta^* \mid v_1^*)}{\sigma^2(\Theta^* \mid v_2^*)}\right]}{1 + \rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(\Theta^* \mid v_2^*)}{\sigma^2(\Theta^* \mid v_1^*)} - 1\right]}}$$
$$+ \frac{\rho(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma(\Theta^* \mid v_2^*)}{\sigma(\Theta^* \mid v_1^*)} - \dfrac{\sigma(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_2^*)}\right]}{\sqrt{1 + \rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(\Theta^* \mid v_2^*)}{\sigma^2(\Theta^* \mid v_1^*)} - 1\right]}}, \tag{5.4.216}$$

where the numerator under the first radical on the right side of the equal sign in (5.4.216) can be removed from under the radical after subtraction. One then has

$$\rho(Z_3,\Theta \mid v_2^*) = \frac{\rho(Z_3,\Theta \mid v_1^*)\dfrac{\sigma(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_2^*)}}{\sqrt{1 + \rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(\Theta^* \mid v_2^*)}{\sigma^2(\Theta^* \mid v_1^*)} - 1\right]}}$$
$$+ \frac{\rho(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma(\Theta^* \mid v_2^*)}{\sigma(\Theta^* \mid v_1^*)} - \dfrac{\sigma(\Theta^* \mid v_1^*)}{\sigma(\Theta^* \mid v_2^*)}\right]}{\sqrt{1 + \rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(\Theta^* \mid v_2^*)}{\sigma^2(\Theta^* \mid v_1^*)} - 1\right]}}, \tag{5.4.217}$$

where the denominators on the right side of the equality are identical, thus permitting the addition of the numerators. After multiplying through the brackets on the second numerator on the right side of the equal sign in (5.4.217), the addition of the numerators yields

$$\rho(Z_3,\Theta \mid v_2^*) = \frac{\rho(Z_3,\Theta \mid v_1^*)\dfrac{\sigma(\Theta^* \mid v_2^*)}{\sigma(\Theta^* \mid v_1^*)}}{\sqrt{1 + \rho^2(Z_3,\Theta \mid v_1^*)\left[\dfrac{\sigma^2(\Theta^* \mid v_2^*)}{\sigma^2(\Theta^* \mid v_1^*)} - 1\right]}}, \tag{5.4.218}$$

where the item associated with $Z_3(v_1^*)$, as it will be aliased, was also eligible to be chosen as the second item in the tailored test. The correlation of this $Z_g(v_1^*)$ with $\Theta^*(v_1^*)$ for that ordering is given by (5.4.40). Thus, unsquared and squared substitutions from (5.4.40) into (5.4.218), while changing the aliased subscript from 2 to 3, allow one to write

$$\rho(Z_3,\Theta \mid v_2^*) = \frac{\dfrac{a_3^*\ \sigma(\Theta^* \mid v_2^*)}{\sqrt{1 + [a_3^*\ \sigma(\Theta^* \mid v_1^*)]^2}}}{\sqrt{1 + \dfrac{[a_3^*\ \sigma(\Theta^* \mid v_2^*)]^2 - [a_3^*\ \sigma(\Theta^* \mid v_1^*)]^2}{1 + [a_3^*\ \sigma(\Theta^* \mid v_1^*)]^2}}}. \tag{5.4.219}$$

After placing the expression under the radical in the lower denominator of (5.4.219) on a common denominator, the subtraction of terms provides

$$\rho(Z_3,\Theta \mid v_2^*) = \frac{\dfrac{a_3^* \, \sigma(\Theta^* \mid v_2^*)}{\sqrt{1 + [a_3^* \, \sigma(\Theta^* \mid v_1^*)]^2}}}{\sqrt{\dfrac{1 + [a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2}{1 + [a_3^* \, \sigma(\Theta^* \mid v_1^*)]^2}}}, \tag{5.4.220}$$

where multiplication of the numerator and denominator by their common denominator yields the result

$$\rho(Z_3,\Theta \mid v_2^*) = \frac{a_3^* \, \sigma(\Theta^* \mid v_2^*)}{\sqrt{1 + [a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2}}. \tag{5.4.221}$$

By definition, one has

$$\rho(Z_3,\Theta \mid v_2^*) = \frac{a_3(v_2^*)}{\sqrt{1 + [a_3(v_2^*)]^2}}, \tag{5.4.222}$$

where $a_3(v_2^*)$ is appropriate for $\tilde{\Theta}^*(v_2^*)$ or $\Theta^*(v_2^*)$ after its standardization to a mean of zero and variance of unity. Upon squaring (5.4.222), one obtains

$$\rho^2(Z_3,\Theta \mid v_2^*) = \frac{[a_3(v_2^*)]^2}{1 + [a_3(v_2^*)]^2} \tag{5.4.223}$$

where substitution from the squared result of (5.4.221), after transposition, provides

$$\frac{[a_3(v_2^*)]^2}{1 + [a_3(v_2^*)]^2} = \frac{[a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2}{1 + [a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2}. \tag{5.4.224}$$

Upon clearing the denominators in (5.4.224), one obtains

$$[a_3(v_2^*)]^2 + [a_3(v_2^*)]^2 \, [a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2 = [a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2 + [a_3(v_2^*)]^2 \, [a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2, \tag{5.4.225}$$

where subtraction of the common expressions on both sides of the equality yields

$$[a_3(v_2^*)]^2 = [a_3^* \, \sigma(\Theta^* \mid v_2^*)]^2. \tag{5.4.226}$$

Taking the square root of (5.4.226), one has

$$a_3(v_2^*) = a_3^* \, \sigma(\Theta^* \mid v_2^*), \tag{5.4.227}$$

indicating that $a_3(v_2^*)$, appropriate for $\tilde{\Theta}^*(v_2^*)$ or $\Theta^*(v_2^*)$ after its standardization to a mean of zero and a variance of unity, is obtained by the multiplication of two knowns, a_3^* and $\sigma(\Theta^* \mid v_2^*)$. Thus (5.4.221) provides a convenient expression for $\rho(Z_3,\Theta \mid v_2^*)$, which was one of the sought solutions.

The Point of Dichotomization on Standardized Z_3 Given the Outcome Vector $\mathbf{v}_2^*$, $\gamma_3(\mathbf{v}_2^*)$

The other solution one now seeks is a convenient expression for $\gamma_3(v_2^*)$, the point of dichotomization on $\tilde{Z}_3(v_2^*)$ or, synonomously, $Z_3(v_2^*)$ after its standardization to a mean of zero and a variance of unity. In obtaining this solution the rescaling of Θ^* for which the parameter $a_3(v_2^*)$ is appropriate will be defined. This definition will lead to a solution for $b_3(v_2^*)$. For completeness, $c_3(v_2^*)$ will then be defined. At this juncture, both $\rho(Z_3,\Theta \mid v_2^*)$ and $b_3(v_2^*)$ will be known. Consequently, $\gamma_3(v_2^*)$ will be known, because $\gamma_3(v_2^*)$ is by definition merely the product of $\rho(Z_3,\Theta \mid v_2^*)$ and $b_3(v_2^*)$. After obtaining this solution the important property of invariance will be further considered.

After *incidental selections* on Θ^* due to *explicit selections* on Z_1 and $\tilde{Z}_2(v_1^*)$, Θ^* can be standardized to a mean of zero and variance of unity. This standardization is accomplished in the usual manner through

$$\tilde{\theta}^*(v_2^*) = \frac{\theta^* - \mu(\Theta^* \mid v_2^*)}{\sigma(\Theta^* \mid v_2^*)}, \tag{5.4.228}$$

where $\tilde{\theta}^*(v_2^*)$ is a realization of the continuous random variable $\tilde{\Theta}^*(v_2^*)$ for which the parameter $a_3(v_2^*)$ is appropriate. The appropriate parameter $b_3(v_2^*)$ is obtained through

$$b_3(v_2^*) = \frac{b_3^* - \mu(\Theta^* \mid v_2^*)}{\sigma(\Theta^* \mid v_2^*)}, \tag{5.4.229}$$

where the expression on the right side of the equality resembles the similarly positioned expression in (5.4.228). This resemblance occurs because, consistent with the definition of the difficulty parameter, $b_3(v_2^*)$ is rendered in the standard scale of ability, in this case that of $\tilde{\Theta}^*(v_2^*)$. The parameters $a_3(v_2^*)$ and $b_3(v_2^*)$ are, as usual, defined identically in both submodels. For the three-parameter normal ogive submodel, one has

$$c_3(v_2') = c_3^* = c_3. \tag{5.4.230}$$

Again as in the case of (3.3.65) and (5.4.49), this is because a change in the scale of ability leaves the lower asymptote of the regression of the binary, multiple-choice item on ability unchanged.

Since $\gamma_3(v_2^*)$, the point of dichotomization on $\tilde{Z}_3(v_2^*)$, is by definition

$$\gamma_3(v_2^*) = \rho(Z_3,\Theta \mid v_2^*)\, b_3(v_2^*), \tag{5.4.231}$$

substitutions from (5.4.221) and (5.4.229) into (5.4.231), along with a cancellation of terms and divisions of the numerator and denominator by a_3^*, now yield

$$\gamma_3(v_2^*) = \frac{b_3^* - \mu(\Theta^* \mid v_2^*)}{[(a_3^*)^{-2} + \sigma^2(\Theta^* \mid v_2^*)]^{.5}}, \tag{5.4.232}$$

the sought solution. In (5.4.232) one has a convenient expression for the point of dichotomization on $\tilde{Z}_3(v_2^*)$ where the required inputs are known.

The Property of Invariance

Remember that the property of invariance pertains to the probability of passing item g. This probability must remain invariant for any particular level, rather than for any particular numerical value, of ability under changes in the scale of ability. Invariance is obtained by simple transformations of the item parameters a_g and b_g given a change in the scale of ability. Earlier, the property of invariance given an arbitrary prescription for the scale of ability was discussed in connection with (2.3.122). Again, in conjunction with (5.4.52), the property of invariance was discussed with respect to a change in the scale of ability resulting from the standardization of Θ^* subsequent to the *incidental selection* on Θ^* imposed by *explicit selection* on Z_1. Here the discussion will be extended to include a change in the scale of ability resulting from the standardization of Θ^* subsequent to the *incidental selections* on Θ^* imposed by *explicit selections* on Z_1 and $\tilde{Z}_2(v_1^*)$. Extending equation (2.3.122), invariance requires the result

$$\gamma_3(\theta) = \gamma_3(\theta^*) = \gamma_3[\tilde{\theta}^*(v_2^*)] \tag{5.4.233}$$

where the third item is potentially any one of the remaining $(p-2)$ unused items in the ability bank. The point of dichotomization $\gamma_3(v_2^*)$ would be the lower limit of integration for the standardized conditional distribution of $\tilde{Z}_3(v_2^*)$ given $\tilde{\theta}^*(v_2^*)$, where the integral yields the probability of producing or recognizing a correct answer to the third item in the tailored test for a specific numerical value of $\tilde{\Theta}^*(v_2^*)$. In this context, $\gamma_3[\tilde{\theta}^*(v_2^*)]$ operates as did $\gamma_3(\theta)$ in the context of (2.1.15). When the level of ability is arbitrarily fixed, $\gamma_3[\tilde{\theta}^*(v_2^*)]$ must equal $\gamma_3(\theta)$ even though the particular numerical values of $\tilde{\Theta}^*(v_2^*)$ and Θ are unequal because of intervening transformations on the scale of ability. Thus the probabilities yielding the item characteristic curve remain undisturbed when the equality in (5.4.233) is maintained. Given invariance, $\gamma_3[\tilde{\theta}^*(v_2^*)]$ will be defined by

$$\gamma_3[\tilde{\theta}^*(v_2^*)] = -\, a_3(v_2^*)\, [\tilde{\theta}^*(v_2^*) - b_3(v_2^*)] \tag{5.4.234}$$

under the necessary condition of equality asserted in (5.4.233). Substitutions from (2.1.15), (2.3.121), and (5.4.234) into (5.4.233) yield

$$-\, a_3(\theta - b_3) = -\, a_3^*(\theta^* - b_3^*) = -\, a_3(v_2^*)[\tilde{\theta}^*(v_2^*) - b_3(v_2^*)] \tag{5.4.235}$$

where substitutions from (5.4.227), (5.4.228), and (5.4.229) into the rightmost member in the equality allows one to write

$$- a_3(\theta - b_3) = - a_3^{\ddagger}(\theta^* - b_3^{\ddagger}) = - a_3^{\ddagger}(\theta^* - b_3^{\ddagger}). \tag{5.4.236}$$

Subsequent substitutions from (2.3.118), (2.3.119), and (2.3.120) into (5.4.236) provide

$$- a_3(\theta - b_3) = - a_3(\theta - b_3) = - a_3(\theta - b_3) \tag{5.4.237}$$

in proof of both the necessary condition which demonstrates the property of invariance and the appropriateness of the parameters $a_3(\nu_3^{\ddagger})$ and $b_3(\nu_3^{\ddagger})$ for the continuous variable of transformed ability $\tilde{\Theta}^*(\nu_3^{\ddagger})$.

6. THE TAILORED TESTING ALGORITHMS IN RECURSIVE FORM

6.1 Some Preliminary Recursive Relationships

In developing the recursive formulation for the tailored testing algorithms it will be helpful to make use of a few conventions in notation. Through the application of these conventions succinct statements can be derived which summarize the major results of previous developments as well as provide important recurring relationships. For instance, the generalized outcome vector, v^*_{n-1}, was first defined in Section 5.2. This vector is defined as containing the binary scores for the answers to the $(n-1)$ previously chosen items along with their original subscripts within the ability bank. In this context, the entry for v^*_{n-1} when n equals *one* represents the condition at the start of testing. Thus, the generalized outcome v^*_0 is a null entry, implied but absent until now. Under this convention, $a_1(v^*_0)$ is identically a_1, appropriate for ability $\tilde{\Theta}^*$, or merely Θ, that is, Θ^* in standard score form. This parameter is defined by (2.2.19) when g is aliased to 1 for the first chosen item, $g^{(1)}$. Accordingly, $a_2(v^*_1)$, appropriate for $\tilde{\Theta}^*(v^*_1)$, is defined in (5.3.47); and $a_3(v^*_2)$, appropriate for $\tilde{\Theta}^*(v^*_2)$, is defined in (5.3.112). In general, then, one will find that $a_n(v^*_{n-1})$, appropriate for $\tilde{\Theta}^*(v^*_{n-1})$, can be written compactly as

$$a_n(v^*_{n-1}) = a^*_n \, \sigma(\Theta^* \mid v^*_{n-1}) \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.1}$$

where it is to be noted that $\sigma(\Theta^* \mid v^*_0)$ is, by convention, merely $\sigma(\Theta^*)$ and q_i is the number of items in the tailored test for individual i. The reader may now review this recursive phenomenon through consulting (2.2.19), (5.3.47), and (5.3.112). Under the convention of generalized notation, one has

$$a_n(v^*_{n-1}) = a_n(v_{n-1}) = a_n(v'_{n-1}) \tag{6.1.2}$$

which is to say that this parameter is defined identically under both submodels.

In similar fashion, $b_1(v^*_0)$, appropriate for $\tilde{\Theta}^*$ or merely Θ, that is, Θ^* in standard score form, is identically b_1, as defined by (2.2.20), when g is aliased to 1 for the first chosen item, $g^{(1)}$. Accordingly, $b_2(v^*_1)$, appropriate for $\tilde{\Theta}^*(v^*_1)$, is defined by (5.3.49); and $b_3(v^*_2)$, appropriate for $\tilde{\Theta}^*(v^*_2)$, is defined by (5.3.114). In general, then, one will find that $b_n(v^*_{n-1})$ can be written compactly as

$$b_n(v^*_{n-1}) = \frac{b^*_n - \mu(\Theta^* \mid v^*_{n-1})}{\sigma(\Theta^* \mid v^*_{n-1})} \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.3}$$

where it is to be noted that $\mu(\Theta^* \mid v^*_0)$ is, by convention, merely $\mu(\Theta^*)$. The reader may now review this recursive phenomenon through consulting (2.2.20), (5.3.49), and (5.3.114). Under the convention of generalized notation, one has, then,

$$b_n(v^*_{n-1}) = b_n(v_{n-1}) = b_n(v'_{n-1}) \tag{6.1.4}$$

which is to say that this parameter is defined identically under both submodels.

Under these conventions, the parameter $c_n(v^*_{n-1})$ can be defined for the two-parameter normal ogive submodel

$$c_n(v_{n-1}) = 0 \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.5}$$

where this parameter is of no consequence in the formulation. Viewed from this perspective, one finds that the two-parameter submodel results when c_g is set equal to zero for all g in the three-parameter submodel or when, in statistical terminology, the three-parameter submodel is degenerate in the parameter c_g. Later this relationship will be used to derive the tailored testing algorithm for the two-parameter submodel from that of the three-parameter submodel by simply setting c_g equal to zero for all g. In the three-parameter normal ogive submodel, one has

$$c_n(v'_{n-1}) = c_n^* = c_n \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.6}$$

as can be verified through consulting (3.2.14)—with n aliased to 1 for the first chosen item, $g^{(1)}$—(5.3.51), and (5.3.116).

Under the conventions, the correlation between $\tilde{Z}_n(v_{n-1}^*)$ and $\tilde{\Theta}^*(v_{n-1}^*)$ is defined by

$$\rho(Z_n, \Theta \mid v_{n-1}^*) = \frac{a_n^* \, \sigma(\Theta^* \mid v_{n-1}^*)}{\{1 + [a_n^* \, \sigma(\Theta^* \mid v_{n-1}^*)]^2\}^{.5}} \quad \text{for } n = 1, 2, \dots q_i, \tag{6.1.7}$$

where this recursive relationship can be confirmed through a review of (2.2.21)—with n aliased to 1 for the first chosen item, $g^{(1)}$—(5.3.48), and (5.3.113). One then has

$$\rho(Z_n, \Theta \mid v_{n-1}^*) = \rho(Z_n, \Theta \mid v_{n-1}) = \rho(Z_n, \Theta \mid v'_{n-1}); \tag{6.1.8}$$

which is to say that this correlation is defined identically under both submodels.

Under the conventions, $\gamma_n(v_{n-1}^*)$, the point of dichotomization on $\tilde{Z}_n(v_{n-1}^*)$, is defined by

$$\gamma_n(v_{n-1}^*) = \frac{b_n^* - \mu(\Theta^* \mid v_{n-1}^*)}{[(a_n^*)^{-2} + \sigma^2(\Theta^* \mid v_{n-1}^*)]^{.5}} \quad \text{for } n = 1, 2, \dots q_i, \tag{6.1.9}$$

which in recursion summarizes (2.2.22)—when n is aliased to 1 for the first chosen item, $g^{(1)}$—(5.3.50), and (5.3.115). One has

$$\gamma_n(v_{n-1}^*) = \gamma_n(v_{n-1}) = \gamma_n(v'_{n-1}) \tag{6.1.10}$$

indicating that the point of dichotomization is defined identically under both submodels. The density in the standard normal distribution at $\gamma_n(v_{n-1}^*)$, $\phi[\gamma_n(v_{n-1}^*)]$, is by definition

$$\phi[\gamma_n(v_{n-1}^*)] = (2\pi)^{-.5} \exp\{-.5\,[\gamma_n(v_{n-1}^*)]^2\} \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.11}$$

where

$$\phi[\gamma_n(v_{n-1}^*)] = \phi[\gamma_n(v_{n-1})] = \phi[\gamma_n(v'_{n-1})] \tag{6.1.12}$$

because of (6.1.10).

The probability of a correct answer to the nth item in generalized notation may be written as

$$P_n^*(v_{n-1}^*) = c_n(v_{n-1}^*) + [1 - c_n(v_{n-1}^*)]\,\Phi[-\gamma_n(v_{n-1}^*)] \quad \text{for } n = 1, 2, \dots q_i. \tag{6.1.13}$$

A substitution from (6.1.6) into (6.1.13) to obtain submodel specificity allows one to write

$$P'_n(v'_{n-1}) = c_n + (1 - c_n)\,\Phi[-\gamma_n(v'_{n-1})] \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.14}$$

for the three-parameter case. In recursion (6.1.14) now summarizes (3.2.1) when n is aliased to 1 for the first selected item $g^{(1)}$, and (5.3.59). For the two-parameter normal ogive submodel, one merely substitutes from (6.1.5) into (6.1.13), while deleting the primes, to obtain

$$P_n(v_{n-1}) = \Phi[-\gamma_n(v_{n-1})] \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.15}$$

which in recursion summarizes (2.2.1)—when n is aliased to 1 for the first chosen item, $g^{(1)}$—and (5.3.57).

Notice that the deletion of the prime on v'_{n-1} in obtaining (6.1.15) merely changes the interpretation of this probability from that of recognizing the correct alternative to a multiple-choice item to that of producing a correct answer to a free-response item. Thus one may write

$$P_n(v'_{n-1}) = \Phi[-\gamma_n(v'_{n-1})] \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.16}$$

for the probability of recognizing the correct alternative given the three-parameter normal ogive submodel.

In generalized notation, the probability of an incorrect answer to the nth item may be written as

$$Q_n^*(v_{n-1}^*) = [1 - c_n(v_{n-1}^*)]\,\Phi[\gamma_n(v_{n-1}^*)] \quad \text{for } n = 1, 2, \dots q_i. \tag{6.1.17}$$

A substitution from (6.1.6) into (6.1.17) to obtain submodel specificity allows one to write

$$Q'_n(v'_{n-1}) = (1 - c_n)\,\Phi[\gamma_n(v'_{n-1})] \quad \text{for } n = 1, 2, \dots q_i \tag{6.1.18}$$

for the three-parameter case. In recursion, (6.1.18) now summarizes (3.2.2) when n is aliased to 1 for the first chosen item $g^{(1)}$, and (5.3.60). As observed in connection with (5.3.65), one has

$$Q_n(v'_{n-1}) = \Phi[\gamma_2(v'_{n-1})] \quad \text{for } n = 1, 2, \ldots q_i \tag{6.1.19}$$

where this probability represents that of not recognizing the correct alternative as opposed to that of obtaining an incorrect answer to the nth item. Because of (6.1.10), this probability is equal to that given by

$$Q_n(v_{n-1}) = \Phi[\gamma_n(v_{n-1})] \tag{6.1.20}$$

which derives from rendering (6.1.17) submodel specific through a substitution from (6.1.5) while deleting the primes. In (6.1.20), one has the probability of producing an incorrect answer to the nth item for the two-parameter case. While (6.1.19) and (6.1.20) produce equal probabilities, their interpretations under each submodel are quite different.

Equations (6.1.13) through (6.1.20) contain an expression in common. This expression is $\Phi[*]$, which is by definition the cumulative normal distribution function or area under the standard normal curve given specific limits of integration from negative infinity to the generalized argument denoted by the asterisk.

6.2 The Tailored Testing Algorithm for the Three-Parameter Normal Ogive Submodel

When one begins testing an individual, it is known that the individual is a member of the population of interest. Nothing else is assumed to be known about this individual. In this uninformed state, the mean of ability in the population of interest, $\mu(\Theta^*)$, is taken as the prior estimate of this individual's ability. Accordingly, the variance or squared error of this estimate is $\sigma^2(\Theta^*)$, the variance or the squared standard deviation of ability in the population of interest. At this point, the parameters a_g^*, b_g^*, and c_g^* are known for each item in the ability bank. As the nth item in this individual's tailored test, that item g is chosen for which

$$\mathscr{E}_{U'_g}\, \sigma^2\, (\Theta^* \mid v'_{n-1}, U'_g) = \sigma^2(\Theta^* \mid v'_{n-1}) \left(1 - \rho^2(Z_g,\Theta \mid v'_{n-1})\,(1 - c_g)\, \frac{\{\phi[\gamma_g(v'_{n-1})]\}^2}{P'_g(v'_{n-1})\, Q_g(v'_{n-1})}\right)$$
$$\text{for all } g \text{ where } g \neq g^{(0)}, g^{(1)}, \ldots g^{(n-1)} \tag{6.2.1}$$

provides the minimum value over the $(p-n+1)$ unused items in the ability bank. Notice that $g^{(0)}$, by convention, places no restriction on the choice of $g^{(n)}$ where $n = 1, 2, \ldots q_i$. The nth chosen item is presented to the individual. One of two outcomes is possible. This individual's answer can be correct or incorrect. If the answer to Item n is correct, the revised or updated estimate of the individual's ability is given by

$$\mu(\Theta^* \mid v'_{n-1}, u'_n = 1) = \mu(\Theta^* \mid v'_{n-1}) + \rho(Z_n,\Theta \mid v'_{n-1})\, \sigma(\Theta^* \mid v'_{n-1})\,(1 - c_n)\, \frac{\phi[\gamma_n(v'_{n-1})]}{P'_n(v'_{n-1})}; \tag{6.2.2}$$

and the variance of this revised or updated estimate of ability is provided by

$$\sigma^2(\Theta^* \mid v'_{n-1}, u'_n = 1)$$
$$= \sigma^2(\Theta^* \mid v'_{n-1}) \left(1 - \rho^2(Z_n,\Theta \mid v'_{n-1})\, \frac{(1 - c_n)\, \phi[\gamma_n(v'_{n-1})]}{P'_n(v'_{n-1})} \left\{\frac{(1 - c_n)\, \phi[\gamma_n(v'_{n-1})]}{P'_n(v'_{n-1})} - \gamma_n(v'_{n-1})\right\}\right). \tag{6.2.3}$$

If the answer to Item n is incorrect, the revised or updated estimate of the individual's ability is given by

$$\mu(\Theta^* \mid v'_{n-1}, u'_n = 0) = \mu(\Theta^* \mid v'_{n-1}) - \rho(Z_n,\Theta \mid v'_{n-1})\, \sigma(\Theta^* \mid v'_{n-1})\, \frac{\phi[\gamma_n(v'_{n-1})]}{Q_n(v'_{n-1})}; \tag{6.2.4}$$

and the variance of this revised or updated estimate of ability is provided by

$\sigma^2(\Theta^* \mid v'_{n-1}, u'_n = 0)$

$$= \sigma^2(\Theta^* \mid v'_{n-1}) \left(1 - \rho^2(Z_n,\Theta \mid v'_{n-1}) \frac{\phi[\gamma_n(v'_{n-1})]}{Q_n(v'_{n-1})} \left\{\frac{\phi[\gamma_n(v'_{n-1})]}{Q_n(v'_{n-1})} + \gamma_n(v'_{n-1})\right\}\right). \tag{6.2.5}$$

Given either outcome, if the error of the estimate, $\sigma(\Theta^* \mid v'_n)$, is equal to or less than the prescribed terminal error, ϵ,

$$\sigma(\Theta^* \mid v'_n) \leq \epsilon, \tag{6.2.6}$$

the testing is terminated. Otherwise the next item n is chosen, and the scenario as outlined for (6.2.1) through (6.2.6) is continued until the condition imposed by (6.2.6) is satisfied. The item satisfying this condition is by definition Item q_i. In application the terminal error, ϵ, is prescribed by the practitioner. This prescription entails considerations that are presented later in Chapter 7.

In proof of (6.2.1) through (6.2.6), it should be noted that the recursive formulation of this algorithm merely represents a summarization and extension of previous results. With respect to previous results: in choosing the item, (6.2.1) summarizes (5.3.30) and (5.3.91); in estimating ability given a correct answer, (6.2.2) summarizes (5.3.39) and (5.3.104); in estimating the variance of ability given a correct answer, (6.2.3) summarizes (5.3.45) and (5.3.109); in estimating ability given an incorrect answer, (6.2.4) summarizes (5.3.41) and (5.3.105); and in estimating the variance of ability given an incorrect answer, (6.2.5) summarizes (5.3.46) and (5.3.110)—for the first two items in the tailored test.

Appropriate substitutions into (6.2.1) through (6.2.5) from the outputs of the preliminary recursive formulation provided in Section 6.1 must be made in evaluating these equations. These substitutions render the inputs for (6.2.1) through (6.2.5) in terms of known values, that is, the mean, the variance, and the item parameters. The density function of the standard normal distribution, $\phi[*]$, and the cumulative normal distribution function, $\Phi[*]$, can then be evaluated.

At this juncture the tailored testing algorithm will be developed for the two-parameter normal ogive submodel. This development can be very brief. Given the tailored testing algorithm presented in Equations (6.2.1) through (6.2.6), it is only necessary to assume that guessing is ineffective. As a consequence, c_g or c_n is zero and the primes can be deleted on all terms. This simplifying assumption leads to the algorithm presented in equations (6.3.1) through (6.3.6). It will then be verified that this simplifying assumption yields a formulation that summarizes the pertinent earlier results. Under the two-parameter normal ogive submodel, the scenario for tailored testing, excluding the formulation, proceeds as it did in the three-parameter case.

6.3. The Tailored Testing Algorithm for the Two-Parameter Normal Ogive Submodel

Again, when one begins testing an individual, it is known that the individual is a member of the population of interest. Nothing else is assumed to be known about this individual. In this uninformed state, the mean of ability in the population of interest, $\mu(\Theta^*)$, is taken as the prior estimate of this individual's ability. Accordingly, the variance or squared error of this estimate is $\sigma^2(\Theta^*)$, the variance or squared standard deviation in the population of interest. At this point, the parameters a^*_g, b^*_g, and c^*_g are known for all the items in the ability bank. Since this submodel is degenerate in the parameter c^*_g, c^*_g is known to be zero for all these items. As the nth item in this individual's tailored test, that item g is chosen for which

$$\mathscr{E}_{U_g}\, \sigma^2(\Theta^* \mid v_{n-1}, U_g) = \sigma^2(\Theta^* \mid v_{n-1}) \left(1 - \rho^2(Z_g,\Theta \mid v_{n-1}) \frac{\{\phi[\gamma_g(v_{n-1})]\}^2}{P_g(v_{n-1})\, Q_g(v_{n-1})}\right)$$

$$\text{for all } g \text{ where } g \neq g^{(0)}, g^{(1)}, \ldots g^{(n-1)} \tag{6.3.1}$$

provides the minimum value over the $(p-n+1)$ unused items in the ability bank. Notice that $g^{(0)}$, by convention, places no restriction on the choice of $g^{(n)}$ where $n = 1, 2, \ldots q_i$. The nth chosen item is presented to the individual. One of two outcomes is possible. This individual's answer can be correct or incorrect. If the answer to Item n is correct, the revised or updated estimate of the individual's ability is given by

$$\mu(\Theta^* \mid v_{n-1}, u_n = 1) = \mu(\Theta^* \mid v_{n-1}) + \rho(Z_n,\Theta \mid v_{n-1})\, \sigma(\Theta^* \mid v_{n-1}) \frac{\phi[\gamma_n(v_{n-1})]}{P_n(v_{n-1})}; \tag{6.3.2}$$

and the variance of this revised or updated estimate of ability is provided by

$$\sigma^2(\Theta^* \mid v_{n-1}, u_n = 1)$$

$$= \sigma^2(\Theta^* \mid v_{n-1}) \left(1 - \rho^2(Z_n,\Theta \mid v_{n-1}) \frac{\phi[\gamma_n(v_{n-1})]}{P_n(v_{n-1})} \left\{\frac{\phi[\gamma_n(v_{n-1})]}{P_n(v_{n-1})} - \gamma_n(v_{n-1})\right\}\right). \tag{6.3.3}$$

If the answer to Item n is incorrect, the revised or updated estimate of the individual's ability is given by

$$\mu(\Theta^* \mid v_{n-1}, u_n = 0) = \mu(\Theta^* \mid v_{n-1}) - \rho(Z_n,\Theta \mid v_{n-1})\, \sigma(\Theta^* \mid v_{n-1}) \frac{\phi[\gamma_n(v_{n-1})]}{Q_n(v_{n-1})}; \tag{6.3.4}$$

and the variance of this revised or updated estimate of ability is provided by

$$\sigma^2(\Theta^* \mid v_{n-1}, u_n = 0)$$

$$= \sigma^2\Theta^* \mid v_{n-1}) \left(1 - \rho^2(Z_n,\Theta \mid v_{n-1}) \frac{\phi[\gamma_n(v_{n-1})]}{Q_n(v_{n-1})} \left\{\frac{\phi[\gamma_n(v_{n-1})]}{Q_n(v_{n-1})} + \gamma_n(v_{n-1})\right\}\right). \tag{6.3.5}$$

Given either outcome, if the error of the estimate, $\sigma(\Theta^* \mid v_n)$, is equal to or less than the prescribed terminal error, ϵ,

$$\sigma(\Theta^* \mid v_n) \leq \epsilon, \tag{6.3.6}$$

the testing is terminated. Otherwise, the next item n is chosen, and the scenario as outlined for (6.3.1) through (6.3.6) is continued until the condition imposed by (6.3.6) is satisfied. The item satisfying this condition is by definition Item q_i.

In application, the terminal error, ϵ, is prescribed by the practitioner. This prescription entails considerations that are presented later in Chapter 7.

In deriving (6.3.1) through (6.3.6) from (6.2.1) through (6.2.6), it is necessary merely to delete the primes on all terms and set c_g or c_n equal to zero. For instance, the cancellation of the primes on $P'_n(v'_{n-1})$ is justified because setting c_n equal to zero in (6.1.14) yields a result that is identical to that given for $P_n(v_{n-1})$ in (6.1.15).

In proof of this algorithm, it will now be verified that (6.3.1) through (6.3.5) summarize the previous results. With respect to previous results: in choosing the item, (6.3.1) summarizes (5.3.18) and (5.3.85); in estimating ability given a correct answer, (6.3.2) summarizes (5.3.35) and (5.3.102); in estimating the variance of ability given a correct answer, (6.3.3) summarizes (5.3.43) and (5.3.107); in estimating ability given an incorrect answer, (6.3.4) summarizes (5.3.37) and (5.3.103); and in estimating the variance of ability given an incorrect answer, (6.3.5) summarizes (5.3.44) and (5.3.108)—for the first two items in the tailored test.

Appropriate substitutions into (6.3.1) through (6.3.5) from the outputs of the preliminary recursive formulation provided earlier in Section 6.1 must be made in evaluating these equations. These substitutions will render the required inputs for (6.3.1) through (6.3.5) in terms of the known values, that is, the mean, the variance, and the item parameters. The density function of the standard normal distribution, $\phi[*]$, and the cumulative normal distribution function, $\Phi[*]$, can then be evaluated.

6.4. A Bayesian Sequential Algorithm

A Bayesian sequential procedure for adaptive or tailored testing was published earlier by Owen (1975). In this publication the one-item situation was considered. The selection of this item was based on minimizing

$$l(d, p, g) = V_0 \left\{1 - (1 - g)(1 + V_0^{-1} p^{-2})^{-1} \frac{[\phi(D)]^2}{A\Phi(D)}\right\}, \tag{6.4.1}$$

the Bayesian preposterior risk. Ability, Θ, in this context was assumed to be distributed normally with a mean M_0 and variance V_0. The parameters d_i, p_i, and g_i were parameters of difficulty, discriminating power, and guessing, respectively. The functions $\phi(D)$ and $\Phi(D)$ were those of the standard normal distribution, its probability density and cumulative distribution functions, respectively, where the argument D was given as

$$D = \frac{(d - M_0)}{\sqrt{p^{-2} + V_0}} \tag{6.4.2}$$

and A was given by

$$A = g + (1 - g)\,\Phi(-D). \tag{6.4.3}$$

The estimator of ability given a correct answer was provided as

$$\mathscr{E}(\Theta \mid 1) = M_0 + (1 - g)\,V_0\,(p^{-2} + V_0)^{-.5}\,\frac{\phi(D)}{A}; \tag{6.4.4}$$

and the estimator for the variance of ability given a correct answer was provided as

$$\operatorname{Var}(\Theta \mid 1) = V_0 \left\{ 1 - (1 - g)\,(1 + p^{-2}V_0^{-1})^{-1}\,\frac{\phi(D)}{A}\left[\frac{(1 - g)\,\phi(D)}{A} - D\right]\right\}. \tag{6.4.5}$$

The estimator of ability given an incorrect answer was provided as

$$\mathscr{E}(\Theta \mid 0) = M_0 - V_0\,(p^{-2} + V_0)^{-.5}\,\frac{\phi(D)}{\Phi(D)}; \tag{6.4.6}$$

and the estimator of the variance of ability given an incorrect answer was provided as

$$\operatorname{Var}(\Theta \mid 0) = V_0 \left\{ 1 - (1 + p^{-2}\,V_0^{-1})^{-1}\,\frac{\phi(D)}{\Phi(D)}\left[\frac{\phi(D)}{\Phi(D)} + D\right]\right\}. \tag{6.4.7}$$

Because this formulation was recursive, Owen suggested that this one-item situation be repeated until V_n, the variance of ability after n items, was acceptably small.

6.5. The Comparison of Algorithms

The parameters p_i, d_i, and g_i in the Owen publication correspond to a_g^*, b_g^*, and c_g^* in this report. Owen's Θ, assumed to be normally distributed with a mean M_0, and variance V_0, corresponds to Θ^* in this report, which is assumed to be normally distributed with a mean, $\mu\Theta^* \mid v_0')$, and variance $\sigma^2(\Theta^* \mid v_0')$. Translating Owen's D, one has

$$D = \frac{b_1^* - \mu(\Theta^* \mid v_0')}{[(a_1^*)^{-2} + \sigma^2(\Theta^* \mid v_0')]^{.5}} \tag{6.5.1}$$

where a substitution from (6.1.9) for n equal to *one* into (6.5.1) yields

$$D = \gamma_1(v_0'). \tag{6.5.2}$$

A translation of Owen's A of (6.4.3), along with substitutions from (6.1.6), provides

$$A = c_1 + (1 - c_1)\,\Phi[-\gamma_1(v_0')] \tag{6.5.3}$$

where a substitution from (6.1.14) for n equal to *one* into (6.5.3) yields

$$A = P_1'(v_0'); \tag{6.5.4}$$

and it is clear after a substitution from (6.5.2) into $\Phi(D)$ that one has

$$\Phi(D) = \Phi[\gamma_1(v_0')] = Q_1(v_0') \tag{6.5.5}$$

because of (6.1.19) when n equals *one*. Through a translation of terms in (6.4.1) and substitutions from (6.5.2), (6.5.4), and (6.5.5) into (6.4.1), one may write the Bayesian preposterior risk as

$$l(b_g^*, a_g^*, c_g^*) = \sigma^2(\Theta^* \mid v_0')\left(1 - \frac{(1 - c_g^*)}{\{1 + [a_g^*\,\sigma(\Theta^* \mid v_0')]^{-2}\}}\,\frac{\{\phi[\gamma_g(v_0')\}^2}{P_g'(v_0')\,Q_g(v_0')}\right) \tag{6.5.6}$$

for the first selected item $g^{(1)}$ where

$$\frac{1}{\{1 + [a_g^*\,\sigma(\Theta^* \mid v_0')]^{-2}\}} = \frac{[a_g^*\,\sigma(\Theta^* \mid v_0')]^2}{1 + [a_g^*\,\sigma(\Theta^* \mid v_0')]^2} = \rho^2(Z_g, \Theta \mid v_0'), \tag{6.5.7}$$

because of (6.1.7) and (6.1.8) when n equals *one*. Substituting from (6.1.6) and (6.5.7) into (6.5.6), one has

$$l(b_g^*, a_g^*, c_g^*) = \sigma^2(\Theta^* \mid v_0') \left(1 - \rho^2(Z_g,\Theta \mid v_0')(1 - c_g) \frac{\{\phi[\gamma_g(v_0')]\}^2}{P_g'(v_0')\, Q_g(v_0')}\right) \tag{6.5.8}$$

or, because of (6.2.1),

$$l(b_g^*, a_g^*, c_g^*) = \mathscr{E}_{U_g'}\, \sigma^2(\Theta^* \mid v_0', U_g'), \tag{6.5.9}$$

the minimum value provided by the first chosen item $g^{(1)}$. Through a translation of terms in (6.4.4) and substitutions from (6.5.2) and (6.5.4), one obtains

$$\mathscr{E}(\Theta^* \mid 1) = \mu(\Theta^* \mid v_0') + \frac{\sigma^2(\Theta^* \mid v_0')}{[(a_1^*)^{-2} + \sigma^2(\Theta^* \mid v_0')]^{.5}} \frac{(1 - c_1^*)\, \phi[\gamma_1(v_0')]}{P_1'(v_0')}, \tag{6.5.10}$$

where multiplication of the numerator and denominator of the first fraction in the rightmost term in (6.5.10) by a_1^*, allows one to write

$$\mathscr{E}(\Theta^* \mid 1) = \mu(\Theta^* \mid v_0') + \frac{a_1^*\, \sigma(\Theta^* \mid v_0')}{\{1 + [a_1^*\, \sigma(\Theta^* \mid v_0')]^2\}^{.5}} \frac{\sigma(\Theta^* \mid v_0')(1 - c_1^*)\, \phi[v_0')]}{P_1'(v_0')}. \tag{6.5.11}$$

One can now substitute from (6.1.6) and (6.1.7), for n equal to *one*, into (6.5.11) to obtain

$$\mathscr{E}(\Theta^* \mid 1) = \mu(\Theta^* \mid v_0') + \rho(Z_1,\Theta \mid v_0')\, \sigma(\Theta^* \mid v_0')(1 - c_1) \frac{\phi[\gamma_1(v_0')]}{P_1'(v_0')} \tag{6.5.12}$$

or

$$\mathscr{E}(\Theta^* \mid 1) = \mu(\Theta^* \mid v_0', u_1' = 1), \tag{6.5.13}$$

because of (6.2.2) when n equals *one*. Through a translation of terms in (6.4.5) and substitutions from (6.5.2) and (6.5.4) into (6.4.5), one may write

$$\operatorname{Var}(\Theta^* \mid 1)$$

$$= \sigma^2(\Theta^* \mid v_0') \left(1 - \frac{(1 - c_1^*)}{\{1 + [a_1^*\, \sigma(\Theta^* \mid v_0')]^{-2}\}} \frac{\phi[\gamma_1(v_0')]}{P_1'(v_0')} \left\{\frac{(1 - c_1^*)\, \phi[\gamma_1(v_0')]}{P_1'(v_0')} - \gamma_1(v_0')\right\}\right), \tag{6.5.14}$$

where substitutions from (6.1.6) and (6.5.7) into (6.5.14) allow one to write

$$\operatorname{Var}(\Theta^* \mid 1)$$

$$= \sigma^2(\Theta^* \mid v_0') \left(1 - \rho^2(Z_1,\Theta \mid v_0') \frac{(1 - c_1)\, \phi[\gamma_1(v_0')]}{P_1'(v_0')} \left\{\frac{(1 - c_1)\, \phi[\gamma_1(v_0')]}{P_1'(v_0')} - \gamma_1(v_0')\right\}\right) \tag{6.5.15}$$

or merely

$$\operatorname{Var}(\Theta^* \mid 1) = \sigma^2(\Theta^* \mid v_0', u_1' = 1) \tag{6.5.16}$$

because of (6.2.3) when n equals *one*. Through a translation of terms in (6.4.6) and substitutions from (6.5.2) and (6.5.5) into (6.4.6), one obtains

$$\mathscr{E}(\Theta^* \mid 0) = \mu(\Theta^* \mid v_0') - \frac{\sigma^2(\Theta^* \mid v_0')}{[(a_1^*)^{-2} + \sigma^2(\Theta^* \mid v_0')]^{.5}} \frac{\phi[\gamma_1(v_0')]}{Q_1(v_0')} \tag{6.5.17}$$

where multiplication of the numerator and denominator of the first fraction in the rightmost term in (6.5.17) by a_1^*, allows one to write

$$\mathscr{E}(\Theta^* \mid 0) = \mu(\Theta^* \mid v_0') - \frac{a_1^*\, \sigma(\Theta^* \mid v_0')}{\{1 + [a_1^*\, \sigma(\Theta^* \mid v_0')]^2\}^{.5}} \frac{\sigma(\Theta^* \mid v_0')\, \phi[\gamma_1(v_0')]}{Q_1(v_0')}. \tag{6.5.18}$$

One can now substitute from (6.1.7), for n equal to *one*, into (6.5.18) to obtain

$$\mathscr{E}(\Theta^* \mid 0) = \mu(\Theta^* \mid v_0') - \rho(Z_1,\Theta \mid v_0')\, \sigma(\Theta^* \mid v_0')\, \frac{\phi[\gamma_1(v_0')]}{Q_1(v_0')} \tag{6.5.19}$$

or

$$\mathscr{E}(\Theta^* \mid 0) = \mu(\Theta^* \mid v_0', u_1' = 0) \tag{6.5.20}$$

because of (6.2.4) when n equals *one*. Through a translation of terms in (6.4.7) and substitutions from (6.5.2) and (6.5.5) into (6.4.7), one may write

$$\mathrm{Var}(\Theta^* \mid 0) = \sigma^2(\Theta^* \mid v_0') \left(1 - \frac{1}{\{1 + [a_1^* \sigma(\Theta^* \mid v_0')]^{-2}\}} \frac{\phi[\gamma_1(v_0')]}{Q_1(v_0')} \left\{\frac{\phi[\gamma_1(v_0')]}{Q_1(v_0')} + \gamma_1(v_0')\right\}\right) \tag{6.5.21}$$

where a substitution from (6.5.7) into (6.5.21) leads to

$$\mathrm{Var}(\Theta^* \mid 0) = \sigma^2(\Theta^* \mid v_0') \left(1 - \rho^2(Z_1,\Theta \mid v_0') \frac{\phi[\gamma_1(v_0')]}{Q_1(v_0')} \left\{\frac{\phi[\gamma_1(v_0')]}{Q_1(v_0')} + \gamma_1(v_0')\right\}\right) \tag{6.5.22}$$

or merely

$$\mathrm{Var}(\Theta^* \mid 0) = \sigma^2(\Theta^* \mid v_0', u_1' = 0) \tag{6.5.23}$$

because of (6.2.5) when n equals *one*.

In the developments culminating in (6.5.9), (6.5.13), (6.5.16), (6.5.20), and (6.5.23), it has been proved that the tailored testing algorithm for the three-parameter normal ogive submodel as based on the selection or rejection analogy is an identity of the Bayesian sequential algorithm provided by Owen. Since both algorithms are recursive, proof of identity in the one-item situation is sufficient. Notice also that the simplifying assumption that guessing is ineffective in the Bayesian case would yield an algorithm for the two-parameter normal ogive submodel. This recursive algorithm would necessarily be an identity of the recursive algorithm provided in Section 6.3 for the two-parameter normal ogive submodel. A demonstration of submodel degeneracy under this assumption was given there.

It is hoped that the exposition in this report based on the selection or rejection analogy has improved the accessibility of the Owen Bayesian algorithm, in particular, the accessibility for psychometricians who are comfortable with selection theory, yet unfamiliar with Bayesian statistics and terminology.

7. THE RELIABILITY OF THE TAILORED TESTING PROCESS

In Chapter 4 it was mentioned that the process of tailored testing produced an individualized test for each examinee. Algorithms were developed for this process in Chapters 5 and 6. This individualized process will be numerically illustrated in Chapter 8. In Chapter 8 it will become apparent that each test is potentially unique. Reliability, of necessity, then must be considered a property of the results of the process rather than a property of a specific test form. This aspect of tailored testing stands in sharp contrast with conventional pencil-and-paper testing where reliability is a property of a specific printed form. In this chapter, an index of reliability for the results of the process of tailored testing will be derived from selection theory. In addition, the implications of this index with respect to the control of the process will be enumerated; and further features of this formulation will be discussed.

The index of reliability represents a binary analogue of the squared multiple correlation. It possesses the following similarities with reliability in classical theory: (a) this index represents the correlation between equiprecise estimates of the same ability which is analogous to the correlation between parallel forms, and (b) the square root of this index represents the correlation between equiprecise ability estimates and true ability, which is analogous to the correlation between test scores and true score.

7.1 The Derivation of an Index of Reliability for the Tailored Testing Process

In this derivation it is assumed that the partial variance of the criterion variable $\Theta^*(v_0^*)$, or merely Θ^*, remains unchanged by *explicit selections* on the partialled variables Z_n where $n = 1, 2, \ldots q_i$. This assumption is algebraically expressed in

$$\sigma^2(\Theta^* \mid v_0^*) - \boldsymbol{\beta}(v_0^*)' \, \Sigma[Zv_0^*),Z(v_0^*)] \, \boldsymbol{\beta}(v_0^*)$$
$$= \sigma^2(\Theta^* \mid v_{q_i}^*) - \boldsymbol{\beta}(v_{q_i}^*)' \, \Sigma[Z(v_{q_i}^*),Z(v_{q_i}^*)] \, \boldsymbol{\beta}(v_{q_i}^*), \tag{7.1.1}$$

where both sides of the equality represent partial variances: on the left, that of $\Theta^*(v_0^*)$ or merely Θ^* before the q_i *explicit selections*, and on the right that of $\Theta^*(v_{q_i}^*)$ after the q_i *explicit selections* on the continuous variables Z_n. In (7.1.1), $\boldsymbol{\beta}(v_0^*)$ is a vector of least squares partial regression coefficients predicting $\Theta^*(v_0^*)$ from the Z_n; and $\boldsymbol{\beta}(v_{q_i}^*)$ is a vector of least squares partial regression coefficients predicting $\Theta^*(v_{q_i}^*)$ from the $Z_n(v_{q_i}^*)$. The variance-covariance matrices, $\Sigma[Z(v_0^*), Z(v_0^*)]$ and $\Sigma[Z(v_{q_i}^*),Z(v_{q_i}^*)]$, are those for the $Z_n(v_0^*)$, or merely the Z_n, before the *explicit selections*, and the $Z_n(v_{q_i}^*)$, the Z_n after q_i *explicit selections*, respectively. The matrix operations on the right side of both equalities in (7.1.1) yield expressions that are familiar from multiple linear regression. From the least squares apparatus one has

$$\boldsymbol{\beta}(v_0^*)' \, \Sigma[Z(v_0^*),Z(v_0^*)] \, \boldsymbol{\beta}(v_0^*) = \sigma^2(\Theta^* \mid v_0^*) \, R^2_{(\Theta^* \cdot Z_1, Z_2, \ldots Z_{qi} \mid v_0^*)} \tag{7.1.2}$$

as the partialled or predicted variance in $\Theta^*(v_0^*)$; and as the partialled or predicted variance in $\Theta^*(v_{q_i}^*)$, one has

$$\boldsymbol{\beta}(v_{q_i}^*)' \, \Sigma[Z(v_{q_i}^*),Z(v_{q_i}^*)] \, \boldsymbol{\beta}(v_{q_i}^*) = \sigma^2(\Theta^* \mid v_{q_i}^*) \, R^2_{(\Theta^* \cdot Z_1, Z_2, \ldots Z_{qi} \mid v_{qi}^*)}. \tag{7.1.3}$$

In (7.1.2) and (7.1.3), the squared multiple correlations, the $R^2_{(*)}$, are those before and after the *explicit selections*, the latter squared multiple correlation being subject to restriction in range. Substitutions from (7.1.2) and (7.1.3) into (7.1.1), along with some rearrangement, lead to

$$\sigma^2(\Theta^* \mid v_0^*) \, [1 - R^2_{(\Theta^* \cdot Z_1, Z_2, \ldots Z_{qi} \mid v_{qi}^*)}] = \sigma^2(\Theta^* \mid v_{q_i}^*) \, [1 - R^2_{(\Theta^* \cdot Z_1, Z_2, \ldots Z_{q_i} \mid v_{q_i}^*)}], \tag{7.1.4}$$

a form that is reminiscent of (5.3.4) and (5.3.55). A solution of (7.1.4) for $\sigma^2(\Theta^* \mid v_{q_i}^*)$, the restricted variance of ability at the termination of the tailored test for individual i, yields

$$\sigma^2(\Theta^* \mid \nu^*_{q_i}) = \sigma^2(\Theta^* \mid \nu^*_0)\,[1 - R^2_{(\Theta^*\cdot Z_1,Z_2,\,\ldots Z_{q_i} \mid \nu^*_0)}] + \sigma^2(\Theta^* \mid \nu^*_{q_i})\, R^2_{(\Theta^*\cdot Z_1,Z_2,\,\ldots Z_{q_i} \mid \nu^*_{q_i})} \tag{7.1.5}$$

where the last term on the right in this equation represents the predictable variance given the full linear regression of Θ^* on the Z_n which is lost through estimating ability from the binary outcomes or item scores. This term may be seen to decrease in value as the number of items in the test increases. An increase in the number of items increases precision or reduces $\sigma^2(\Theta^* \mid \nu^*_{q_i})$; and continued *explicit selections* on the Z_n result in a more severe restriction in range, which is accompanied by a reduction in the squared multiple correlation, $R^2_{(\Theta^*\cdot Z_1,Z_2,\,\ldots Z_{q_i} \mid \nu^*_{q_i})}$. The important implication here is that with properly configured ability banks the full power of multiple linear regression can be approximated by estimating ability from binary outcomes or item scores. Notice that the output of (7.1.5) approaches, as n increases, the squared standard error of estimate, the first term on the right side of (7.1.5), because the numerical value of the last term on the right side of (7.1.5) becomes negligible.

In (7.1.5) the within-individual variance has been examined. Knowledge of the expected value of this variance over individuals will allow us to solve for

$$\eta^2 = \frac{\sigma^2(\Theta^* \mid \nu^*_0) - \mathscr{E}_i\, \sigma^2(\Theta^* \mid \nu^*_{q_i})}{\sigma^2(\Theta^* \mid \nu^*_0)} \tag{7.1.6}$$

because $\sigma^2(\Theta^* \mid \nu^*_0)$ is known. The quantity provided by (7.1.6) is commonly known as the correlation ratio. In this context it indexes a curvilinear relationship unless the $\sigma^2(\Theta^* \mid \nu^*_{q_i})$ are held constant across individuals. When the tailoring process is begun, linearity of regressions and homoscedastic partial variances are assumed. If the process terminates with homoscedastic partial variances, the regression of the ability estimates will be linear. Termination of the tailored tests while satisfying the condition of equiprecision, guarantees homoscedastic partial variances about the linear regression of the ability estimates. There is something novel here. An ability estimate is a point on the line of regression of $\Theta^*(\nu^*_0)$, or merely Θ^*, on the Z_n for each individual i. Equiprecision at the termination of testing over the individuals i, where $i = 1,2,\ldots N$, renders the N linear regressions collinear. The ability estimates for all individuals will then form a linear regression. Given a linear regression, one has

$$\rho^2[\Theta,\mu(\Theta^* \mid \nu^*_{q_i})] = \frac{\sigma^2(\Theta^* \mid \nu^*_0) - \mathscr{E}_i \sigma^2(\Theta^* \mid \nu^*_{q_i})}{\sigma^2(\Theta^* \mid \nu^*_0)} = \frac{\sigma^2(\Theta^* \mid \nu^*_0) - \epsilon^2}{\sigma^2(\Theta^* \mid \nu^*_0)} \tag{7.1.7}$$

as the squared linear correlation between the θ^* and the ability estimates, the $\mu(\Theta^* \mid \nu^*_{q_i})$. In the rightmost equality of (7.1.7), the constant variance at the termination of tailored testing replaces the expectation in the centered equality, because the expectation of a constant is the constant, in this case the terminal variance ϵ^2. Equation (7.1.7) may also be written as

$$\rho[\mu(\Theta^* \mid \nu^*_{q_i}),\mu(\Theta^* \mid \nu^*_{r_i})] = \frac{\sigma^2(\Theta^* \mid \nu^*_0) - \epsilon^2}{\sigma^2(\Theta^* \mid \nu^*_0)} \tag{7.1.8}$$

the linear correlation between distinct ability estimates, the $\mu(\Theta^* \mid \nu^*_{q_i})$, and the $\mu(\Theta^* \mid \nu^*_{r_i})$ of equiprecision. In (7.1.7) and (7.1.8) one has an index of reliability for the results obtained through the tailoring process. As n increases and the last term on the right in (7.1.5) vanishes, one then has

$$\begin{aligned} \epsilon^2 &= \mathscr{E}_i\, \sigma^2(\Theta^* \mid \nu^*_0)\,[1 - R^2_{(\Theta^*\cdot Z_1,Z_2,\,\ldots Z_{q_i} \mid \nu^*_0)}] \\ &= \sigma^2(\Theta^* \mid \nu^*_0)\,[1 - R^2_{(\Theta^*\cdot Z_1,Z_2,\,\ldots Z_{q_i} \mid \nu^*_0)}], \end{aligned} \tag{7.1.9}$$

because the expectation of a constant is the constant. Upon substituting (7.1.9) into (7.1.7) and its identity (7.1.8), one may write

$$\rho^2(\Theta,\mu(\Theta^* \mid \nu^*_{q_i})] = \rho[\mu(\Theta^* \mid \nu^*_{q_i}),\mu(\Theta^* \mid \nu^*_{r_i})] = R^2_{(\Theta^*\cdot Z_1,Z_2,\,\ldots Z_{q_i} \mid \nu^*_0)} \tag{7.1.10}$$

as the upper bound to this index of reliability when the q_i, or the q_i and the r_i are increased indefinitely. This upper bound represents the squared multiple correlation, constant for sets of Z_n over individuals, that is obtainable using the full multiple linear regression equation for the regression of $\Theta^*(\nu^*_0)$, or merely Θ^* on the q_i or the r_i Z_n for all individuals.

7.2. The Index of Reliability and Its Implications for Practice

The terminal variance or its square root ϵ, the terminal error, can be prescribed. It is a constant in the tailoring process. As a constant in the process, the prescription must occur in advance, because ϵ is as applicable to the first as it is to the last tailored test. The variance of Θ^*, $\sigma^2(\Theta^* \mid v_0^*)$, is fixed or known at beginning of the tailoring process. Consequently, a prescribed ϵ determines the outputs of (7.1.7) and (7.1.8), the reliability of the results obtained through the tailoring process. A prescription for ϵ is, therefore, a specification for the reliability of the results obtained through the tailoring process. One can then explicitly solve (7.1.7) for ϵ given a desired lower limit to the reliability as

$$\epsilon = \sigma(\Theta^* \mid v_0^*) \sqrt{1 - \rho^2[\Theta,\mu(\Theta^* \mid v_{q_i}^*)]} \tag{7.2.1}$$

and use the output of (7.2.1) as the terminal error in either (6.2.6) or (6.3.6)—depending on the submodel involved.

In practice, tailored tests are terminated when $\sigma(\Theta^* \mid v_n^*)$ is at most ϵ or when

$$\sigma(\Theta^* \mid v_n^*) \leq \epsilon \tag{7.2.2}$$

first obtains. As a result, the outputs of (7.1.7) and (7.1.8) represent a lower bound to the actual reliability of the results obtained through the process. The error in estimating ability is actually less than that used in obtaining the lower bound to the reliability as given by (7.1.7) and (7.1.8). The actual reliability approaches the lower bound as ϵ becomes more stringent because the item-to-item decrement in $\sigma(\Theta^* \mid v_n^*)$ becomes less and less as finite n in the tailored test increases. Thus, ϵ is more closely approached at termination. For typical ranges of reliability, say, from .8 and greater, actual reliability will closely approach this lower bound.

Notice that (7.1.7) provides a population parameter. The actual correlation between Θ^* and $\mu(\theta^* \mid v_{q_i}^*)$ in *random samples* from the population of interest is an estimate of the parameter that has the output of (7.1.7) as its lower bound. It should be pointed out that the output of (7.1.7) has little relevance to the actual correlation between Θ^* and $\mu(\Theta^* \mid v_{q_i}^*)$ in *biased samples* with respect to the population of interest. This situation, of course, points out the critical requirements of defining the population of interest prior to parameter estimation and of maintaining that definition in the tailoring situation.

The critical requirements of accurate estimates for the item parameters and of an ability bank with a proper configuration of item parameters were discussed earlier in Chapter 4. These critical requirements must be met in order to effectively practice tailored testing as advanced in this report.

In the empirical study alluded to earlier (Urry, 1977), the actual reliabilities, after taking attenuation in the paper-and-pencil test into account, exceeded the lower bounds to the reliability for some eight levels of ϵ. In this study, these lower bounds ranged from .70 to .93. This particular empirical study, therefore, provides convincing evidence that the basic assumptions of the three-parameter normal ogive submodel and the tailoring algorithm for the three-parameter normal ogive submodel can be reasonably well met with extant multiple-choice items.

There are several desirable consequences ensuing from the method of tailored test termination described in connection with (6.2.6) and (6.3.6). These consequences should be noticed: a comprehensive evaluation of all basic assumptions is enabled; process reliability is rendered controllable; equiprecision in ability estimation, an essential component in test fairness, is monitored; and subsequent applications of tailored test results are rendered more tractable because linear relationships have been maintained.

This method of tailored test termination can be extended to the case where several ability estimates are obtained for each individual.These several estimates are derived through the tailoring of several tests given the items from each of several well-conditioned ability banks. This extension of the method, its basic assumptions, and its practical value will be discussed in a subsequent report. In brief, the value of this extended method resides in providing the bases for the emergence of tailored testing systems. A basis is provided for: the communication of users' needs to the tailored testing system; and the monitoring of the tailored testing system with respect to how well these needs are being met. In other words, the products of the tailored testing system can be defined; and the quality of these products can be monitored.

8. A NUMERICAL ILLUSTRATION OF A TAILORED TEST

For the purpose of illustration, the tailored testing algorithm for the three-parameter normal ogive submodel will be used. This algorithm is currently more relevant given the available data bases. In this illustration, the number of items in the ability bank has been restricted to twenty. This restriction would be unrealistic in applications, but is very convenient for the purpose of illustration. It also reduces the computational effort, making the mathematical problem more manageable for those who might wish to benefit from practice with the numerical process. As a further unrealistic but convenient restriction, the actual text of the items in the ability bank will not be considered.

8.1. Initialization of the Process

Before testing can begin, the item parameters for the p items in the ability bank must be known; the scale of measurement for ability must be prescribed, and a terminal error, ϵ, must be determined. In practice, the item parameters must be estimated. The item parameters appropriate for Θ where the mean of Θ, $\mu(\Theta)$, and the variance of Θ, $\sigma^2(\Theta)$, are zero and unity, respectively, have been estimated in a large data set where the numbers of items and individuals were both sizable. The number of items for the ability bank was subsequently restricted to twenty. The parameter estimates $\hat{a}_g$, $\hat{b}_g$, and $\hat{c}_g$ are presented in Table 8.1.1. It has been decided that the mean of the ability estimates should be 100 and the variance of these estimates should be 400. This decision was based on a requirement to render the ability estimates in a familiar scale with a mean of 100 and a standard deviation of 20. There is also a requirement that the reliability of the resulting ability estimates be .90. This also means that the correlation between the ability estimates and true ability will be .95. In order to fulfill the joint requirements, $\mu(\Theta^* \mid v_0')$ will have to be set equal to 100; and $\sigma^2(\Theta^* \mid v_0')$ will have to be set equal to 444.4444. The estimates of ability are regressed estimates, thus the variance of Θ^* must be set such that

$$\sigma^2[\mu(\Theta^* \mid v_{q_i}')] = \rho^2[\Theta,\mu(\Theta^* \mid v_{q_i}')]\ \sigma^2(\Theta^* \mid v_0') \qquad (8.1.1)$$

where $\sigma^2[\mu(\Theta^* \mid v_{q_i}')]$ is the desired variance of the estimates of ability and $\rho^2[\Theta,\mu(\Theta^* \mid v_{q_i}')]$ is the lower bound to the reliability, in this case 400 and .90, respectively. The numerical solution for the required $\sigma^2(\Theta^* \mid v_0')$ is then 444.4444 where the standard deviation, $\sigma(\Theta^* \mid v_0')$, is 21.0819. The appropriate terminal error, ϵ, is provided by (7.2.1) as 6.6667. At this level of prescribed reliability, the actual reliability of the ability estimates obtained through the tailoring process will closely approach the lower bound of .90. Thus, the obtained mean and variance will approach the desired mean and standard deviation in large samples from the population of interest. A complete discussion of tailored testing reliability was given in Chapter 7.

At this juncture, one has prescribed $\mu(\Theta^* \mid v_0')$, $\sigma^2(\Theta^* \mid v_0')$, and, consequently, $\sigma(\Theta^* \mid v_0')$. Also, ϵ, the terminal error, has been calculated. The parameters a_g^*, b_g^*, and c_g^* that are appropriate given the prescription can now be obtained. The use of the conventional notation for the estimates, that is, $\hat{a}_g$, $\hat{b}_g$, and $\hat{c}_g$, will be discontinued. In doing this it is tacitly assumed that the estimates of the parameters presented in Table 8.1.1 are sufficiently accurate for the population of interest. The appropriate parameters are presented in Table 8.1.2. These parameters are obtained through the invariance transformations given in (2.2.17), (2.2.18), and (3.2.14) for the a_g^*, the b_g^*, and the c_g^*.

8.2. The Testing of the Individual

As the testing of this individual is begun, it is known that the individual is a member of the population of interest. Thus the prior, least squares estimate of ability is $\mu(\Theta^* \mid v_0')$ or 100; and the prior estimate of the variance of this ability estimate, $\sigma^2(\Theta^* \mid v_0')$, is 444.4444. The testing of this individual will continue until the error of the estimate of ability, $\sigma(\Theta^* \mid v_{q_i}')$, becomes equal to or less than ϵ or 6.6667.

TABLE 8.1.1

Estimated Parameters for the Items in the Ability Bank Given Θ Where μ(Θ) Equals Zero and $\sigma^2(\Theta)$ Equals Unity

Item (g)	Estimated Parameters $\hat{a}_g$	$\hat{b}_g$	$\hat{c}_g$
1	1.96	−1.92	.16
2	2.36	−1.71	.23
3	1.88	−1.50	.21
4	2.41	−1.29	.22
5	2.10	−1.09	.19
6	2.16	−.92	.17
7	2.06	−.74	.22
8	2.33	−.49	.15
9	1.85	−.28	.16
10	2.61	−.11	.11
11	2.09	.12	.14
12	2.50	.31	.22
13	2.19	.51	.13
14	2.07	.69	.20
15	1.80	.93	.14
16	2.23	1.08	.21
17	1.82	1.32	.13
18	2.53	1.53	.20
19	1.97	1.70	.16
20	2.28	1.94	.18

$p = 20$

Item 1 can now be chosen. In Table 8.2.1 a numerical summary is presented of the process resulting in the choosing of Item 1. For convenience in exposition, the numerical summaries for the choosing of each of the items in this tailored test will have a similar structure. For purposes of clarity, the conventions of recursive notation introduced in Section 6.1 will be observed in these numerical summaries. In these summaries and for columns (2) through (11) of the table, the equation involved in producing the tabled entries will be identified. In column (1), the item's subscript within the ability bank is presented. Columns (2), (3), and (4) provide the parameters $a_g(\nu_0')$, $b_g(\nu_0')$, and c_g appropriate for standardized $\Theta^*(\nu_0')$, by convention $\bar{\Theta}^*(\nu_0')$, or merely Θ. These parameters are obtained from (6.1.1), (6.1.3), and (6.1.6) for n equal to 1. The required inputs for these equations are a_g^*, b_g^*, and c_g^* as given in Table (8.1.2), and $\mu(\Theta^* \mid \nu_0')$ and $\sigma(\Theta^* \mid \nu_0')$ or 100 and 21.0819, respectively. Within the rounding error introduced in Table 8.1.2, these parameters are identical to the corresponding estimates in Table 8.1.1. Sufficient accuracy has been assumed for these estimates. In column (5), one has the correlation between $\bar{Z}_g(\nu_0')$ and $\Theta^*(\nu_0')$, or merely Z_g and $\Theta^*, \rho(Z_g, \Theta \mid \nu_0')$, as provided by (6.1.7) for n equal to 1; because each item in the ability bank is, as of now, potentially Item 1. The point of dichotomization on $\bar{Z}_g(\nu_0')$, $\gamma_g(\nu_0')$, is given in column (6). It is obtained most readily by the row-wise multiplication of columns (3) and (5), which, within rounding error, results in the identical output of (6.1.9) under the circumstance that Item 1 is being sought. The density in the standard normal distribution at $\gamma_g(\nu_0')$, $\phi[\gamma_g(\nu_0')]$, is presented in column (7). The density is obtained from (6.1.11). The probability that this individual will recognize the correct answer to multiple-choice item g, $P_g(\nu_0')$, is presented in column (8) and is obtained from (6.1.16) under the circumstance that Item 1 is sought. Column (9) contains the probability that this individual will not recognize the correct alternative to multiple-choice item g, $Q_g(\nu_0')$. This probability is obtained most readily as the row-wise complement of column (8), that is, $[1 - P_g(\nu_0')]$, which is identically the output of (6.1.19) under this circumstance. Column (10) presents the probability of obtaining a correct answer to multiple-choice item g, $P_g'(\nu_0')$. These numerical values are obtained from (6.1.14) for n equal to 1 where each item in the ability bank is potentially Item 1. In column (11) one has the expected values of the variance of the ability estimate, $\mathscr{E}_{U_g'}\, \sigma^2(\Theta^* \mid \nu_0', U_g')$, given the possible outcomes for each item. Each expected value is obtained from (6.2.1) using $\sigma^2(\Theta^* \mid \nu_0')$ or 444.4444 and the row-wise entries in columns (4), (5), (7), (9), and (10) as inputs. Given the entries in column (11), the most informative item is sought, that is, the item which, when responded to by this individual, is expected to reduce most the variance of the ability estimate. As mentioned earlier, this reduction increases the precision of the ability estimate or, analogously, most restricts, in expectation, the variance of ability for that subpopulation

TABLE 8.1.2
Parameters for the Items in the Ability Bank Given Θ^* Where $\mu(\Theta^*)$ and $\sigma^2(\Theta^*)$ Have Been Prescribed

Item (g)	Parameters a_g^*	b_g^*	c_g^*
1	.09297	59.5228	.16
2	.11194	63.9500	.23
3	.08918	68.3772	.21
4	.11432	72.8044	.22
5	.09961	77.0208	.19
6	.10246	80.6047	.17
7	.09771	84.3994	.22
8	.11052	89.6699	.15
9	.08775	94.0971	.16
10	.12380	97.6810	.11
11	.09914	102.5298	.14
12	.11859	106.5354	.22
13	.10388	110.7517	.13
14	.09819	114.5465	.20
15	.08538	119.6061	.14
16	.10578	122.7684	.21
17	.08633	127.8280	.13
18	.12001	132.2552	.20
19	.09345	135.8391	.16
20	.10815	140.8988	.18

Note. $\mu(\Theta^*) = 100.00$; $\sigma^2(\Theta^*) = 444.4444$; $p = 20$.

which would have been selected or rejected had *explicit selection* actually occurred on the continuous variable Z_1. In column (12) it is indicated that the item subscripted 10 within the ability bank is to have 1 as its subscript n within the tailored test. As can be noted in column (11), this item provided the minimum expected value for the variance of the ability estimate given the p items in the ability bank. This minimum expected value was 244.4, thus identifying Item 1 in the tailored test as Item 10 from the ability bank. According to convention, then, $g^{(1)}$ equals 10.

Table 8.2.1
Numerical Summary for the Choosing of Item 1

(1) Item (g)	(2) $a_g(v_0')$ (6.1.1)[a]	(3) $b_g(v_0')$ (6.1.3)	(4) c_g (6.1.6)	(5) $\rho(Z_g,\Theta \mid v_0')$ (6.1.7)	(6) $\gamma_g(v_0')$ (6.1.9)	(7) $\phi[\gamma_g(v_0')]$ (6.1.11)	(8) $P_g(v_0')$ (6.1.16)	(9) $Q_g(v_0')$ (6.1.19)	(10) $P_g'(v_0')$ (6.1.14)	(11) $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_0',U_g')$ (6.2.1)	(12) n
1	1.9600	−1.9200	.16	.8908	−1.7103	.0924	.9564	.0436	.9634	384.2	
2	2.3599	−1.7100	.23	.9207	−1.5745	.1155	.9423	.0577	.9556	374.2	
3	1.8801	−1.5000	.21	.8829	−1.3243	.1660	.9073	.0927	.9268	356.7	
4	2.4101	−1.2900	.22	.9236	−1.1915	.1962	.8833	.1167	.9090	337.2	
5	2.1000	−1.0900	.19	.9029	−.9841	.2458	.8375	.1625	.8684	318.8	
6	2.1600	−.9200	.17	.9075	−.8349	.2816	.7981	.2019	.8324	301.2	
7	2.0599	−.7400	.22	.8996	−.6657	.3197	.7472	.2528	.8028	303.2	
8	2.3300	−.4900	.15	.9189	−.4503	.3605	.6737	.3263	.7227	268.6	
9	1.8499	−.2800	.16	.8797	−.2463	.3870	.5973	.4027	.6617	282.1	
10	2.6099	−.1100	.11	.9338	−.1027	.3968	.5409	.4591	.5914	244.4	1
11	2.0901	.1200	.14	.9021	.1082	.3966	.4569	.5431	.5329	275.4	
12	2.5001	.3100	.22	.9285	.2878	.3828	.3867	.6133	.5217	307.6	
13	2.1900	.5100	.13	.9097	.4639	.3582	.3214	.6786	.4096	296.7	
14	2.0700	.6900	.20	.9004	.6213	.3289	.2672	.7328	.4138	341.6	
15	1.8000	.9300	.14	.8742	.8130	.2867	.2081	.7919	.3190	349.4	
16	2.2300	1.0800	.21	.9125	.9855	.2455	.1622	.8378	.3381	382.3	
17	1.8200	1.3200	.13	.8764	1.1569	.2043	.1237	.8763	.2376	384.9	
18	2.5300	1.5300	.20	.9300	1.4229	.1450	.0774	.9226	.2619	417.7	
19	1.9701	1.7000	.16	.8917	1.5159	.1265	.0648	.9352	.2144	420.8	
20	2.2800	1.9400	.18	.9158	1.7766	.0823	.0378	.9622	.2110	434.2	

[a]Numbers in parentheses indicate the equations providing the column entries.

This individual then responds with a correct answer to Item 1. Thus the outcome vector v_1' contains the entry $u_{1(10)}' = 1$, where the tenth item in the ability bank has been aliased by the subscript 1 within the tailored test. Because this individual's answer to Item 1 was correct, the estimate of ability can be revised by using (6.2.2) where the required inputs are $\mu(\Theta^* \mid v_0')$, or 100, $\sigma(\Theta^* \mid v_0')$, or 21.0819, and the unrounded values corresponding to the entries in Table 8.2.1 from columns (4), (5), (7), and (10), for g equal to 10, the subscript of the aliased item. In this case, the evaluation of (6.2.2) yields 111.7567 as the updated estimate of this individual's ability, $\mu(\Theta^* \mid v_1')$. Accordingly, the variance of this estimate of ability can be revised through the use of (6.2.3), where the required inputs are $\sigma^2(\Theta^* \mid v_1')$ or 444.4444, and the unrounded values corresponding to the entries in Table 8.2.1 from columns (4), (5), (6), (7),and (10) for g equal to 10, the subscript of the aliased item. In this case, the evaluation of (6.2.3) provides 282.4501 as the updated variance of this estimate of ability, $\sigma^2(\Theta^* \mid v_1')$. The error of this estimate, $\sigma(\Theta^* \mid v_1')$, is therefore 16.8063. Since 16.8063 is greater than 6.6667 with respect to the comparison in (6.2.6), which does not satisfy the condition stipulated by (6.2.6), the testing is continued.

Item 2 can now be chosen. In Table 8.2.2 a numerical summary is presented for the process resulting in the choosing of Item 2. Notice that there are no entries in this table for $g^{(1)}$ or g equal to 10, because Item $g^{(1)}$ has already been used in this tailored test. In column (1) the subscripts of the items in the ability bank are presented. Columns (2), (3), and (4) provide the parameters $a_g(v_1')$, $b_g(v_1')$, and c_g. These parameters are appropriate for $\tilde{\Theta}^*(v_1')$ or standardized ability subsequent to the incidental effect of explicit selection on $\tilde{Z}_{1(10)}$ as specified in the outcome vector v_1'. The parameters are obtained from (6.1.1), (6.1.3), and (6.1.6) for n equal to 2 where each of the unused items is potentially Item 2. The required inputs for these equations are a_g^*, b_g^*, and c_g^* as given in Table 8.1.2, and $\mu(\Theta^* \mid v_1')$ and $\sigma(\Theta^* \mid v_1')$ or 111.7567 and 16.8063, respectively. Column (5) presents the correlation between $Z_g(v_1')$ and $\Theta^*(v_1')$, $\rho(Z_g,\Theta \mid v_1')$, as provided by (6.1.7) for n equal to 2; because each unused item in the ability bank is potentially Item 2. The point of dichotomization on $\tilde{Z}_g(v_1')$—standardized $Z_g(v_1')$ after the effect of incidental selection due to explicit selection on $\tilde{Z}_{1(10)}$—, $\gamma_g(v_1')$, is given in column (6). This point of dichotomization is obtained most readily by the row-wise multiplication of columns (3) and (5), which, within rounding error, results in the identical output of (6.1.9) under the circumstance that Item 2 is being sought. The density in the standard normal distribution at $\gamma_g(v_1')$, $\phi[\gamma_g(v_1')]$, is given in column (7). This density is obtained by evaluating (6.1.11) for n equal to 2. The probability that this individual will recognize the correct answer to multiple-choice item g, $P_g(v_1')$, is provided in column (8) and is obtained from (6.1.16) given the circumstance that Item 2 is sought. Column (9) presents the probability that this individual will not recognize the correct alternative to multiple-choice item g, $Q_g(v_1')$. This probability is obtained most

Table 8.2.2
Numerical Summary for the Choosing of Item 2

(1) Item (g)	(2) $a_g(v_1')$ (6.1.1)[a]	(3) $b_g(v_1')$ (6.1.3)	(4) c_g (6.1.6)	(5) $\rho(Z_g,\Theta \mid v_1')$ (6.1.7)	(6) $\gamma_g(v_1')$ (6.1.9)	(7) $\phi[\gamma_g(v_1')]$ (6.1.11)	(8) $P_g(v_1')$ (6.1.16)	(9) $Q_g(v_1')$ (6.1.19)	(10) $P_g'(v_1')$ (6.1.14)	(11) $\mathscr{E}_{U_g}\sigma^2(\Theta^* \mid v_1',U_g')$ (6.2.1)	(12) n
1	1.5625	−3.1080	.16	.8423	−2.6178	.0130	.9956	.0044	.9963	276.0	
2	1.8813	−2.8446	.23	.8830	−2.5118	.0170	.9940	.0060	.9954	274.2	
3	1.4988	−2.5812	.21	.8318	−2.1471	.0398	.9841	.0159	.9874	266.9	
4	1.9213	−2.3177	.22	.8870	−2.0559	.0482	.9801	.0199	.9845	261.9	
5	1.6741	−2.0668	.19	.8585	−1.7744	.0826	.9620	.0380	.9692	251.2	
6	1.7220	−1.8536	.17	.8648	−1.6029	.1104	.9455	.0545	.9548	241.4	
7	1.6421	−1.6278	.22	.8541	−1.3903	.1518	.9178	.0822	.9359	234.3	
8	1.8574	−1.3142	.15	.8805	−1.1572	.2042	.8764	.1236	.8949	212.3	
9	1.4747	−1.0508	.16	.8277	−.8697	.2733	.8078	.1922	.8385	207.1	
10											1
11	1.6662	−.5490	.14	.8574	−.4707	.3571	.6811	.3189	.7257	184.1	
12	1.9931	−.3107	.22	.8938	−.2777	.3839	.6094	.3906	.6953	187.0	
13	1.7458	−.0598	.13	.8677	−.0519	.3984	.5207	.4793	.5830	177.4	2
14	1.6502	.1660	.20	.8552	.1420	.3949	.4436	.5564	.5548	199.0	
15	1.4349	.4671	.14	.8204	.3832	.3707	.3508	.6492	.4417	204.1	
16	1.7778	.6552	.21	.8716	.5711	.3389	.2840	.7160	.4343	219.8	
17	1.4509	.9563	.13	.8234	.7874	.2926	.2155	.7845	.3175	225.2	
18	2.0169	1.2197	.20	.8959	1.0928	.2196	.1373	.8627	.3098	249.7	
19	1.5705	1.4329	.16	.8435	1.2087	.1922	.1134	.8866	.2552	254.9	
20	1.8176	1.7340	.18	.8762	1.5192	.1258	.0644	.9356	.2328	269.5	

[a]Numbers in parentheses indicate equations providing the column entries.

readily as the row-wise complement of column (8), that is $[1 - P_g(v_1')]$, which is identically the output of (6.1.19) for this circumstance. Column (10) presents the probability that this individual will obtain a correct answer to multiple-choice item g, $P_g'(v_1')$. These numerical values are obtained by evaluating (6.1.14) for n equal to 2 where each unused item in the ability bank is potentially Item 2. In column (11) one has the expected values of the variance of this individual's ability estimate, $\mathscr{E}_{U_g'}\, \sigma^2(\Theta^* \mid v_1', U_g')$, given the probable outcomes for each unused item. Each expected value is obtained from (6.2.1) using $\sigma^2(\Theta^* \mid v_1')$ or 282.4501 and the row-wise entries in columns (4), (5), (7), (9), and (10) as inputs. The minimum entry in column (11) is found in order to identify Item 2, the most informative second item for the tailored test. When responded to by this individual, Item 2 is expected to reduce most the variance of the ability estimate or, synonomously, to provide the maximum increase in precision given the available items. The minimum value of 177.4 identifies Item 13 from the ability bank as Item 2 in the tailored test. As a result, a 2 is recorded in column 12 and row 13 of Table 8.2.2. According to convention, then, $g^{(2)}$ equals 13. Under the selection or rejection analogy, explicit selection or rejection on $\tilde{Z}_{2(13)}$ subsequent to explicit selection on $\tilde{Z}_{1(10)}$ is expected to restrict most the variance of ability for the selected or rejected subpopulation if *explicit selection* actually occurred in this sequence on $\tilde{Z}_2(v_1')$.

This individual then responds with a correct answer to Item 2. Thus, the outcome vector v_2' contains the entries $u_{1(10)}' = 1$ and $u_{2(13)}' = 1$ where the tenth and thirteenth items in the ability bank have been aliased by the subscripts 1 and 2, respectively, within the tailored test. Because this individual's answer to Item 2 was correct, the estimate of ability can be revised using (6.2.2) where the required inputs are $\mu(\Theta^* \mid v_1')$ or 111.7567, $\sigma(\Theta^* \mid v_1')$ or 16.8063, and the unrounded values corresponding to the entries in Table 8.2.2 from columns (4), (5), (7), and (10) for g equal to 13, the subscript of the second aliased item. In this case, the evaluation of (6.2.2) provides 120.4269 as the updated estimate of this individual's ability, $\mu(\Theta^* \mid v_2')$. Accordingly, the variance of this estimate of ability can be revised by evaluating (6.2.3) where the required inputs are $\sigma^2(\Theta^* \mid v_1')$ or 282.4501 and the unrounded values corresponding to the entries in Table 8.2.2 from columns (4), (5), (6), (7), and (10) for n equal to 13, the subscript of the second aliased item. In this case, the use of (6.2.3) yields 200.7159 as the updated variance of this estimate of ability, $\sigma^2(\Theta^* \mid v_2')$. The error of this estimate, $\sigma(\Theta^* \mid v_2')$, is 14.1674. Since 14.1674 is greater than 6.6667 with respect to the comparison given in (6.2.6), which does not satisfy the condition stipulated in (6.2.6), testing is continued.

The third item can now be chosen. In Table 8.2.3, a numerical summary is presented for the process resulting in the choosing of Item 3. Notice that there are no entries in this table for $g^{(1)}$ or g equal to 10 and $g^{(2)}$ or g equal to 13. The items with these subscripts are ineligible for subsequent use in this tailored test. In column (1), the subscripts for the items in the ability bank are given. Columns (2), (3), and (4) present the parameters

Table 8.2.3
Numerical Summary for the Choosing of Item 3

(1) Item (g)	(2) $a_g(v_2')$ (6.1.1)[a]	(3) $b_g(v_2')$ (6.1.3)	(4) c_g (6.1.6)	(5) $\rho(Z_g,\Theta \mid v_2')$ (6.1.7)	(6) $\gamma_g(v_2')$ (6.1.9)	(7) $\phi[\gamma_g(v_2')]$ (6.1.11)	(8) $P_g(v_2')$ (6.1.16)	(9) $Q_g(v_2')$ (6.1.19)	(10) $P_g'(v_2')$ (6.1.14)	(11) $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_2', U_g')$ (6.2.1)	(12) n
1	1.3171	−4.2989	.16	.7965	−3.4239	.0011	.9997	.0003	.9997	200.3	
2	1.5859	−3.9864	.23	.8459	−3.3720	.0014	.9996	.0004	.9997	200.2	
3	1.2635	−3.6739	.21	.7841	−2.8808	.0063	.9980	.0020	.9984	198.8	
4	1.6196	−3.3614	.22	.8509	−2.8602	.0067	.9979	.0021	.9983	198.3	
5	1.4112	−3.0638	.19	.8159	−2.4998	.0175	.9938	.0062	.9950	195.3	
6	1.4516	−2.8108	.17	.8235	−2.3147	.0274	.9897	.0103	.9914	192.4	
7	1.3843	−2.5430	.22	.8106	−2.0614	.0477	.9804	.0196	.9847	188.6	
8	1.5658	−2.1710	.15	.8428	−1.8297	.0748	.9663	.0337	.9714	180.0	
9	1.2432	−1.8585	.16	.7792	−1.4481	.1398	.9262	.0738	.9380	171.8	
10											1
11	1.4046	−1.2633	.14	.8146	−1.0291	.2349	.8483	.1517	.8695	152.8	
12	1.6801	−.9805	.22	.8593	−.8426	.2797	.8003	.1997	.8442	147.1	
13											2
14	1.3911	−.4151	.20	.8120	−.3370	.3769	.6319	.3681	.7056	142.8	3
15	1.2096	−.0579	.14	.7707	−.0447	.3985	.5178	.4822	.5853	143.0	
16	1.4986	.1653	.21	.8318	.1375	.3952	.4453	.5547	.5618	145.7	
17	1.2231	.5224	.13	.7742	.4044	.3676	.3429	.6571	.4284	150.5	
18	1.7002	.8349	.20	.8620	.7196	.3079	.2359	.7641	.3887	162.6	
19	1.3239	1.0879	.16	.7980	.8681	.2737	.1927	.8073	.3218	169.8	
20	1.5322	1.4450	.18	.8374	1.2101	.1918	.1131	.8869	.2728	183.2	

[a]Numbers in parentheses indicate the equations providing the column entries.

$a_g(v_2')$, $b_g(v_2')$, and c_g for these eligible items. These parameters are appropriate for $\tilde{\Theta}^*(v_2')$ or standardized ability subsequent to the incidental effects of explicit selections on $\tilde{Z}_{1(10)}(v_0')$, and $\tilde{Z}_{2(13)}(v_1')$ as specified in the outcome vector v_2'. The parameters are obtained from (6.1.1), (6.1.3) and (6.1.6) for n equal to 3 where each of the eligible items is potentially Item 3. The required inputs for these equations are a_g^*, b_g^*, and c_g^* as provided in Table 8.1.2, and $\mu(\Theta^* \mid v_2')$ and $\sigma(\Theta^* \mid v_2')$ or 120.4269 and 14.1674, respectively. In column (5), one has the correlation between $Z_g(v_2')$ and $\Theta^*(v_2')$, $\rho(Z_g,\Theta \mid v_2')$, as given by (6.1.7) because each unused item in the ability bank is potentially Item 3. The point of dichotomization on the continuous variable $\tilde{Z}_g(v_2')$—standardized $Z_g(v_2')$ after the effects of incidental selection due to the explicit selections on $\tilde{Z}_{1(10)}(v_0')$ and $\tilde{Z}_{2(13)}(v_1')$—, $\gamma_g(v_2')$, is given in column (6). This point of dichotomization is most conveniently obtained by the row-wise multiplication of columns (3) and (5) which, within rounding error, results in the identical output of (6.1.9) under the circumstance that Item 3 is being sought. The density in the standard normal distribution at $\gamma_g(v_2')$, $\phi[\gamma_g(v_2')]$, is provided in column (7). This density is obtained by evaluating (6.1.11) for n equal to 3. The probability that this individual will recognize the correct alternative to multiple-choice item g, $P_g(v_2')$, is presented in column (8) and is obtained from (6.1.16) given the circumstance that Item 3 is sought. Column (9) provides the probability that this individual will not recognize the correct alternative to multiple-choice item g, $Q_g(v_2')$. This probability is obtained most readily as the row-wise complement of column (8), that is $[1 - P_g(v_2')]$, which is identically the output of (6.1.19) for this circumstance. Column (10) provides the probability that this individual will obtain a correct answer to multiple-choice item g, $P_g'(v_2')$. These numerical values are obtained by evaluating (6.1.14) for n equal to 3 where each unused item in the ability bank is potentially Item 3. In column (11) one has the expected values of the variance of this individual's ability estimate, $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_2',U_g')$, given the probable outcomes for each unused item. Each expected value is obtained from (6.2.1) using $\sigma^2(\Theta^* \mid v_2')$ or 200.7159 and the row-wise entries in columns (4), (5), (7), (9), and (10) as inputs. The minimum entry in column (11) is found in order to identify Item 3, the most informative third item for this tailored test. When responded to by this individual, Item 3 is expected to most reduce the variance of the ability estimate or, synonomously, to provide the maximum increase in its precision. The minimum value of 142.8 identifies Item 14 from the ability bank as Item 3 in this tailored test. As a result, a 3 is recorded in column (12) and row 14 of Table 8.2.3. Under the selection or rejection analogy, explicit selection or rejection on $\tilde{Z}_{3(14)}(v_2')$ subsequent to the explicit selection on $\tilde{Z}_{1(10)}(v_0')$ and $\tilde{Z}_{2(13)}(v_1')$ is expected to restrict most the variance of ability for the selected or rejected subpopulation if *explicit selection* actually occurred in this sequence on $\tilde{Z}_{3(14)}(v_2')$.

This individual then responds with an incorrect answer to Item 3. Thus, the outcome vector v_3' now contains the entries $u_{1(10)}' = 1$, $u_{2(13)}' = 1$, $u_{3(14)}' = 0$ where the tenth, thirteenth, and fourteenth items in the ability bank have been aliased by the subscripts 1, 2, and, 3, respectively, within this tailored test. Because this individual's answer to Item 3 was incorrect, the estimate of ability can be revised by evaluating (6.2.4), where the required inputs are $\mu(\Theta^* \mid v_2')$ or 120.4269, $\sigma(\Theta^* \mid v_2')$ or 14.1674, and the entries in Table 8.2.3 from columns (5), (7), and (9) for g equal to 14, the subscript of the third aliased item. In this case, the evaluation of (6.2.4) yields 108.6462 as the updated estimate of this individual's ability, $\mu(\Theta^* \mid v_3')$. Accordingly, the variance of this estimate of ability can be revised through the use of (6.2.5) where the required inputs are $\sigma^2(\Theta^* \mid v_2')$ or 200.7159 and the unrounded values corresponding to the entries in Table 8.2.3 from columns (5), (6), (7), and (9) for g equal to 14, the subscript of the third aliased item. In this case, the use of (6.2.5) provides 107.6038 as the updated variance of the estimate of ability, $\sigma^2(\Theta^* \mid v_3')$. The error of this estimate, $\sigma(\Theta^* \mid v_3')$, is 10.3732. Since 10.3732 is greater than 6.6667 with regard to the comparison given in (6.2.6), which does not satisfy the stipulated condition of (6.2.6), the testing is continued.

The fourth item can now be chosen. In Table 8.2.4, a numerical summary is provided for the process leading to the choosing of Item 4. Notice that there are no entries in this table for $g^{(1)}$ or g equal to 10, $g^{(2)}$ or g equal to 13, and $g^{(3)}$ or g equal to 14. These items are ineligible for subsequent use in this tailored test. In column (1), the subscripts for the items in the ability bank are presented. Columns (2), (3), and (4) provide the parameters $a_g(v_3')$, $b_g(v_3')$, and c_g for these eligible items. These parameters are appropriate for $\tilde{\Theta}^*(v_3')$ or standardized ability subsequent to the incidental effects of the *explicit selections* on $\tilde{Z}_{1(10)}(v_0')$, $\tilde{Z}_{2(13)}(v_1')$, and $\tilde{Z}_{3(14)}(v_2')$ as specified in the outcome vector v_3'. The parameters are obtained from (6.1.1), (6.1.3), and (6.1.6) for n equal to 4 where each of the eligible items is potentially Item 4. The required inputs for these equations are a_g^*, b_g^*, and c_g^* as provided in Table 8.1.2, $\mu(\Theta^* \mid v_3')$ or 108.6462 and $\sigma(\Theta^* \mid v_3')$ or 10.3732. In column (5), one has the correlation between $Z_g(v_3')$ and $\Theta^*(v_3')$, $\rho(Z_g,\Theta \mid v_3')$, as given by (6.1.7), because each unused item in the ability bank is potentially Item 4. The point of dichotomization on the continuous variable $\tilde{Z}_g(v_3')$—standardized $Z_g(v_3')$ after the effects of incidental selection due to the *explicit selections* on $\tilde{Z}_{1(10)}(v_0')$, $\tilde{Z}_{2(13)}(v_1')$, and $\tilde{Z}_{3(14)}(v_2')$—, $\gamma_g(v_3')$, is given in column (6). This point of dichotomization is most readily obtained by the

Table 8.2.4
Numerical Summary for the Choosing of Item 4

(1) Item (g)	(2) $a_g(v_3')$ (6.1.1)[a]	(3) $b_g(v_3')$ (6.1.3)	(4) c_g (6.1.6)	(5) $\rho(Z_g,\Theta \mid v_3')$ (6.1.7)	(6) $\gamma_g(v_3')$ (6.1.9)	(7) $\phi[\gamma_g(v_3')]$ (6.1.11)	(8) $P_g(v_3')$ (6.1.16)	(9) $Q_g(v_3')$ (6.1.19)	(10) $P_g'(v_3')$ (6.1.14)	(11) $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_3',U_g')$ (6.2.1)	(12) n
1	.9644	−4.7356	.16	.6942	−3.2873	.0018	.9995	.0005	.9996	107.3	
2	1.1612	−4.3088	.23	.7577	−3.2649	.0019	.9995	.0005	.9996	107.3	
3	.9251	−3.8820	.21	.6791	−2.6362	.0124	.9958	.0042	.9967	106.2	
4	1.1859	−3.4552	.22	.7645	−2.6414	.0122	.9959	.0041	.9968	105.8	
5	1.0333	−3.0488	.19	.7186	−2.1908	.0362	.9858	.0142	.9885	103.4	
6	1.0628	−2.7033	.17	.7283	−1.9688	.0574	.9755	.0245	.9797	101.1	
7	1.0136	−2.3374	.22	.7119	−1.6639	.0999	.9519	.0481	.9625	98.4	
8	1.1464	−1.8294	.15	.7536	−1.3786	.1542	.9160	.0840	.9286	91.8	
9	.9103	−1.4026	.16	.6731	−.9441	.2555	.8274	.1726	.8551	89.5	
10											1
11	1.0284	−.5896	.14	.7169	−.4227	.3648	.6638	.3362	.7108	81.1	
12	1.2302	−.2035	.22	.7760	−.1579	.3940	.5627	.4373	.6589	80.4	4
13											2
14											3
15	.8857	1.0566	.14	.6630	.7005	.3121	.2418	.7582	.3480	92.6	
16	1.0973	1.3614	.21	.7391	1.0062	.2405	.1572	.8428	.3342	98.1	
17	.8955	1.8492	.13	.6671	1.2336	.1864	.1087	.8913	.2245	100.4	
18	1.2449	2.2760	.20	.7796	1.7744	.0826	.0380	.9620	.2304	106.0	
19	.9694	2.6215	.16	.6960	1.8246	.0755	.0340	.9660	.1886	106.2	
20	1.1219	3.1092	.18	.7465	2.3210	.0270	.0101	.9899	.1883	107.4	

[a] Numbers in parentheses indicate the equations providing the column entries.

row-wise multiplication of the entries in columns (3) and (5), an operation which, within rounding error, provides an output identical to that of (6.1.9) under the circumstance that Item 4 is being sought. The density in the standard normal distribution at $\gamma_g(v_3')$, $\phi[\gamma_g(v_3')]$, is presented in column (7). This density is obtained by evaluating (6.1.11) for n equal to 4. The probability that this individual will recognize the correct alternative to multiple-choice item g, $P_g(v_3')$, is presented in column (8) and is obtained from (6.1.16) under the circumstance that Item 4 is sought. Column (9) provides the probability that this individual will not recognize the correct alternative to multiple-choice item g, $Q_g(v_3')$. This probability is obtained most conveniently as the row-wise complement of column (8), that is $[1 - P_g(v_3')]$, which results in the identical output of (6.1.19) for this circumstance. Column (10) presents the probability that this individual will obtain a correct answer to multiple-choice item g, $P_g'(v_3')$. In obtaining these numerical values, (6.1.14) is evaluated for n equal to 4 where each unused item in the ability bank is potentially Item 4. Column (11) provides the expected values of the variance of this individual's ability estimate, $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_3',U_g')$, given the probable outcomes for each unused item in the ability bank. Each expected value is obtained by evaluating (6.2.1) where the required inputs are $\sigma^2(\Theta^* \mid v_3')$ or 107.6038 and the row-wise entries in columns (4), (5), (7), (9), and (10). The minimum entry in column (11) is found in order to identify Item 4, the most informative fourth item for this tailored test. When responded to by this individual, Item 4 is expected to reduce most the variance of the ability estimate given the eligible items. Synonomously, Item 4 is expected to provide the maximum increase in precision, given the available items. The minimum value of 80.4 identifies Item 12 from the ability bank as Item 4 in this tailored test. Therefore, a 4 is recorded in column 12 and row 12 of Table 8.2.4. Under the selection or rejection analogy, explicit selection or rejection on $\tilde{Z}_{4(12)}(v_3')$ subsequent to explicit selections on $\tilde{Z}_{1(10)}(v_0')$ and $\tilde{Z}_{2(13)}(v_1')$ and explicit rejection on $\tilde{Z}_{3(14)}(v_2')$ is expected to restrict most the variance of ability for the selected or rejected subpopulation if *explicit selection* actually occurred in this sequence on $\tilde{Z}_{4(12)}(v_3')$.

This individual then responds with an incorrect answer to Item 4. Therefore, the outcome vector v_4' contains the entries $u_{1(10)}' = 1$, $u_{2(13)}' = 1$, $u_{3(14)}' = 0$), and $u_{4(12)}' = 0$ where the tenth, thirteenth, fourteenth, and twelfth items in the ability bank have been aliased by the subscripts 1, 2, 3, and 4, respectively, within this individual's tailored test. Because this individual's answer to Item 4 was incorrect, the estimate of ability can be revised by evaluating (6.2.4) where the required inputs are $\mu(\Theta^* \mid v_3')$ or 108.6462, $\sigma(\Theta^* \mid v_3')$ or 10.3732, and the unrounded values corresponding to the entries in Table 8.2.4 from columns (5), (7), and (9) for g equal to 12, the subscript of the fourth aliased item. In this case, the evaluation of (6.2.4) yields 101.3935 as the updated estimate of this individual's ability, $\mu(\Theta^* \mid v_4')$. Accordingly, the variance of this estimate of ability can be revised through the use of (6.2.5) where the required inputs are $\sigma^2(\Theta^* \mid v_3')$ or 107.6038 and the unrounded values corresponding to the entries in Table 8.2.4 from columns (5), (6), (7), and (9) for n equal to 12, the subscript of the fourth aliased item. In this case, the use of (6.2.5) provides 64.2195 as the updated variance of this estimate of ability, $\sigma^2(\Theta^* \mid v_4')$. The error of this estimate, $\sigma(\Theta^* \mid v_4')$, is 8.0137. Since 8.0137 is greater than 6.6667 with regard to the comparison given in (6.2.6), which does not satisfy the condition stipulated in (6.2.6), the testing of this individual is continued.

The fifth item can now be chosen. In Table 8.2.5, a numerical summary is presented for the process leading to the choosing of Item 5. Notice that there are no entries in this table for $g^{(1)}$ or g equal to 10, $g^{(2)}$ or g equal to 13, $g^{(3)}$ or g equal to 14, and $g^{(4)}$ or g equal to 12. These items are ineligible for subsequent use in this tailored test. In column (1), the subscripts for the items in the ability bank are presented. The parameters $a_g(v_4')$, $b_g(v_4')$ and c_g are presented in columns (2), (3), and (4) for the eligible items. These parameters are appropriate for $\bar{\Theta}^*(v_4')$ or standardized ability subsequent to the incidental effects of *explicit selections* on $\bar{Z}_{1(10)}(v_0')$, $\bar{Z}_{2(13)}(v_1')$, $\bar{Z}_{3(14)}(v_2')$, and $\bar{Z}_{4(12)}(v_3')$ as specified in the outcome vector v_4'. The parameters are obtained from (6.1.1), (6.1.3), and (6.1.6) for n equal to 5 where each of the unused or eligible items is potentially Item 5. The required inputs for these equations are a_g^*, b_g^*, and c_g^* as provided in Table 8.1.2, $\mu(\Theta^* \mid v_4')$ or 101.3935 and $\sigma(\Theta^* \mid v_4')$ or 8.0137. In column (5), one has the correlation between $Z_g(v_4')$ and $\Theta^*(v_4')$, $\rho(Z_g, \Theta \mid v_4')$, as given by (6.1.7) because as of now each unused item in the ability bank is potentially Item 5. The point of dichotomization on the continuous variable $\bar{Z}_g(v_4')$—standardized $Z_g(v_4')$ given the effects of incidental selection due to *explicit selections* on $\bar{Z}_{1(10)}(v_0')$, $\bar{Z}_{2(13)}(v_1')$, $\bar{Z}_{3(14)}(v_2')$, and $\bar{Z}_{4(12)}(v_3')$—, $\gamma_g(v_4')$, is provided in column (6). This point of dichotomization is most readily obtained by the row-wise multiplication of the entries in columns (3) and (5), an operation that, within rounding error, provides an output identical to that of (6.1.9) under the circumstance that Item 5 is being sought. The density in the standard normal distribution at $\gamma_g(v_4')$, $\phi[\gamma_g(v_4')]$, is provided in column (7). This density is obtained by evaluating (6.1.11) for n equal to 5. The probability that this individual will recognize the correct alternative to multiple-choice item g, $P_g(v_4')$ is given in column (8) and is obtained from (6.1.16) under the circumstance that Item 5 is sought. Column (9) provides the probability that this individual will not recognize the correct alternative to multiple-choice item g, $Q_g(v_4')$. This probability is obtained most conveniently as the row-wise complement of column (8), that is $[1 - P_g(v_4')]$, which results in the identical output of (6.1.19) for this circumstance. Column (10) provides the probability that this individual will obtain a correct answer to multiple-choice item g, $P_g'(v_4')$. In obtaining these numerical values, (6.1.14) is evaluated for n equal to 5 where each eligible item in the ability bank is potentially Item 5. In column (11), one has the expected values of the variance of this individual's ability estimate, $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_4', U_g')$, given the probable outcomes for each eligible item. Each expected value is obtained by evaluating (6.2.1) where the required inputs

Table 8.2.5
Numerical Summary for the Choosing of Item 5

(1) Item (g)	(2) $a_g(v_4')$ (6.1.1)[a]	(3) $b_g(v_4')$ (6.1.3)	(4) c_g (6.1.6)	(5) $\rho(Z_g,\Theta \mid v_4')$ (6.1.7)	(6) $\gamma_g(v_4')$ (6.1.9)	(7) $\phi[\gamma_g(v_4')]$ (6.1.11)	(8) $P_g(v_4')$ (6.1.16)	(9) $Q_g(v_4')$ (6.1.19)	(10) $P_g'(v_4')$ (6.1.14)	(11) $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_4',U_g')$ (6.2.1)	(12) n
1	.7450	−5.2249	.16	.5974	−3.1216	.0031	.9991	.0009	.9992	64.0	
2	.8971	−4.6724	.23	.6678	−3.1200	.0031	.9991	.0009	.9993	64.0	
3	.7147	−4.1200	.21	.5814	−2.3955	.0226	.9917	.0083	.9934	63.2	
4	.9161	−3.5675	.22	.6755	−2.4099	.0219	.9920	.0080	.9938	62.8	
5	.7982	−3.0414	.19	.6239	−1.8974	.0659	.9711	.0289	.9766	61.1	
6	.8211	−2.5942	.17	.6346	−1.6462	.1029	.9501	.0499	.9586	59.5	
7	.7830	−2.1206	.22	.6165	−1.3074	.1697	.9045	.0955	.9255	58.0	
8	.8857	−1.4629	.15	.6630	−.9700	.2492	.8340	.1660	.8589	53.8	
9	.7032	−.9105	.16	.5752	−.5237	.3478	.6998	.3002	.7478	54.6	
10											1
11	.7945	.1418	.14	.6221	.0882	.3974	.4649	.5351	.5398	52.5	5
12											4
13											2
14											3
15	.6842	2.2727	.14	.5647	1.2833	.1751	.0997	.9003	.2257	61.6	
16	.8477	2.6673	.21	.6466	1.7247	.0901	.0423	.9577	.2434	63.5	
17	.6918	3.2987	.13	.5689	1.8767	.0686	.0303	.9697	.1563	63.7	
18	.9617	3.8511	.20	.6932	2.6695	.0113	.0038	.9962	.2030	64.2	
19	.7489	4.2983	.16	.5994	2.5765	.0144	.0050	.9950	.1642	64.2	
20	.8667	4.9297	.18	.6549	3.2287	.0022	.0006	.9994	.1805	64.2	

[a]Numbers in parentheses indicate the equations providing the column entries.

are $\sigma^2(\Theta^* \mid v_4')$ or 64.2195 and the row-wise entries in columns (4), (5), (7), (9), and (10). The minimum entry in column (11) is sought in order to identify the most informative fifth item for this tailored test. The minimum value of 52.5 identifies Item 11 from the ability bank as Item 5 in this tailored test. Thus, a 5 is recorded in column 12 and row 11 of Table 8.2.5. Under the selection or rejection analogy, explicit selection or rejection on $\bar{Z}_{5(11)}(v_4')$ subsequent to explicit selections on $\bar{Z}_{1(10)}(v_0')$ and $\bar{Z}_{2(13)}(v_1')$ and explicit rejections on $\bar{Z}_{3(14)}(v_2')$ and $\bar{Z}_{4(12)}(v_3')$ is expected to restrict most the variance of ability for the selected or rejected subpopulation if *explicit selection* on $\bar{Z}_{5(11)}(v_4')$ actually occurred in this sequence.

This individual then responds with an incorrect answer to Item 5. Hence, the outcome vector v_5' contains the entries $u_{1(10)}' = 1$, $u_{2(13)}' = 1$, $u_{3(14)}' = 0$, $u_{4(12)}' = 0$, and $u_{5(11)}' = 0$ where the tenth, thirteenth, fourteenth, twelfth, and eleventh items in the ability bank have been aliased by the subscripts 1, 2, 3, 4, and 5, respectively, within this tailored test. Because this individual's answer to Item 5 was incorrect, the estimate of ability can be revised by evaluating (6.2.4) where the required inputs are $\mu(\Theta^* \mid v_4')$ or 101.3935, $\sigma(\Theta^* \mid v_4')$ or 8.0137, and the unrounded values corresponding to the entries in Table 8.2.5 from columns (5), (7) and (9) for g equal to 11, the subscript of the fifth aliased item. In this case, the evaluation of (6.2.4) yields 97.6917 as the updated estimate of this individual's ability, $\mu(\Theta^* \mid v_5')$. Accordingly, the variance of this estimate of ability can be revised through the use of (6.2.5) where the required inputs are $\sigma^2(\Theta^* \mid v_4')$ or 64.2195 and the unrounded values corresponding to the entries in Table 8.2.5 from columns (5), (6), (7), and (9) for n equal to 11, the subscript of the fifth aliased item. In this case, the use of (6.2.5) yields 48.8885 as the updated variance of this estimate of ability, $\sigma^2(\Theta^* \mid v_5')$. The error of this estimate, $\sigma(\Theta^* \mid v_5')$, is 6.9920. Since 6.9920 is greater than 6.6667 with respect to the comparison given in (6.2.6), which does not satisfy the condition stipulated in (6.2.6), the testing is continued.

The sixth item can now be chosen. In Table 8.2.6, a numerical summary is provided for the process resulting in the choosing of Item 6. Notice that there are no entries in this table for $g^{(1)}$ or g equal to 10, $g^{(2)}$ or g equal to 13, $g^{(3)}$ or g equal to 14, $g^{(4)}$ or g equal to 12, and $g^{(5)}$ or g equal to 11. These items are ineligible for subsequent use in this tailored test. In column (1), the subscripts for the eligible as well as the ineligible items in the ability bank are presented. The parameters $a_g(v_5')$, $b_g(v_5')$ and c_g for the eligible items are provided in columns (2), (3), and (4). These parameters are appropriate for $\bar{\Theta}^*(v_5')$ or standardized ability subsequent to the incidental effects of *explicit selections* on $\bar{Z}_{1(10)}(v_0')$, $\bar{Z}_{2(13)}(v_1')$, $\bar{Z}_{3(14)}(v_2')$, $\bar{Z}_{4(12)}(v_3')$, and $\bar{Z}_{5(11)}(v_4')$ as specified in the outcome vector v_5'. The parameters are obtained from (6.1.1), (6.1.3), and (6.1.6) for n equal to 6 where each of the unused or eligible items is potentially Item 6. The required inputs for these equations are a_g^*, b_g^*, and c_g^* as provided in Table 8.1.2, $\mu(\Theta^* \mid v_5')$ or 97.6917, and $\sigma(\Theta^* \mid v_5')$ or 6.9920. In column (5), the correlation

Table 8.2.6
Numerical Summary for the Choosing of Item 6

(1) Item (g)	(2) $a_g(v_5')$ (6.1.1)[a]	(3) $b_g(v_5')$ (6.1.3)	(4) c_g (6.1.6)	(5) $\rho(Z_g,\Theta \mid v_5')$ (6.1.7)	(6) $\gamma_g(v_5')$ (6.1.9)	(7) $\phi[\gamma_g(v_5')]$ (6.1.11)	(8) $P_g(v_5')$ (6.1.16)	(9) $Q_g(v_5')$ (6.1.19)	(10) $P_g'(v_5')$ (6.1.14)	(11) $\mathscr{E}_{U_g'}\sigma^2(\Theta^* \mid v_5', U_g')$ (6.2.1)	(12) n
1	.6500	−5.4589	.16	.5450	−2.9752	.0048	.9985	.0015	.9988	48.7	
2	.7827	−4.8257	.23	.6163	−2.9743	.0048	.9985	.0015	.9989	48.7	
3	.6235	−4.1926	.21	.5291	−2.2183	.0341	.9867	.0133	.9895	47.9	
4	.7993	−3.5594	.22	.6244	−2.2224	.0338	.9869	.0131	.9898	47.6	
5	.6965	−2.9564	.19	.5715	−1.6896	.0957	.9544	.0456	.9631	46.2	
6	.7164	−2.4438	.17	.5824	−1.4232	.1449	.9227	.0773	.9358	44.9	
7	.6832	−1.9011	.22	.5641	−1.0724	.2245	.8582	.1418	.8894	44.0	
8	.7728	−1.1473	.15	.6115	−.7015	.3119	.7585	.2415	.7947	41.0	6
9	.6136	−.5141	.16	.5230	−.2689	.3848	.6060	.3940	.6690	42.6	
10											1
11											5
12											4
13											2
14											3
15	.5970	3.1342	.14	.5126	1.6066	.1098	.0541	.9459	.1865	48.1	
16	.7396	3.5865	.21	.5946	2.1327	.0410	.0165	.9835	.2230	48.8	
17	.6036	4.3101	.13	.5168	2.2273	.0334	.0130	.9870	.1413	48.8	
18	.8391	4.9433	.20	.6428	3.1775	.0026	.0007	.9993	.2006	48.9	
19	.6534	5.4558	.16	.5470	2.9843	.0046	.0014	.9986	.1612	48.9	
20	.7562	6.1795	.18	.6032	3.7272	.0004	.0001	.9999	.1801	48.9	

[a] Numbers in parentheses indicate the equations providing the column entries.

between $Z_g(v_5')$ and $\Theta^*(v_5')$, $\rho(Z_g,\Theta \mid v_5')$, is presented. This correlation is obtained from (6.1.7) for n equal to 6 because each unused item in the ability bank is potentially Item 6. The point of dichotomization on the continuous variable $\bar{Z}_g(v_5')$—standardized $Z_g(v_5')$ after the effects of incidental selection due to *explicit selections* on $\bar{Z}_{1(10)}(v_0')$, $\bar{Z}_{2(13)}(v_1')$, $\bar{Z}_{3(14)}(v_2')$, $\bar{Z}_{4(12)}(v_3')$, and $\bar{Z}_{5(11)}(v_4')$—, $\gamma_g(v_5')$, is provided in column (6). This point of dichotomization can be conveniently obtained through the row-wise multiplication of the entries in columns (3) and (5). This multiplication provides, within rounding error, an output that is equal to that of (6.1.9) given the circumstance that Item 6 is being sought. The density of the standard normal distribution at $\gamma_g(v_5')$, $\phi[\gamma_g(v_5')]$, is presented in column (7). This density is obtained through evaluating (6.1.11) for n equal to 6. The probability that this individual will recognize the correct alternative to multiple-choice item g, $P_g(v_5')$, is given in column (8). This probability can be obtained from (6.1.16) given that Item 6 is sought. The probability that this individual will not recognize the correct alternative to multiple-choice item g, $Q_g(v_5')$, is provided in column (9). This probability can be most readily obtained as the row-wise complement of column (8), that is, $[1 - P_g(v_5')]$. This complement is identically the output of (6.1.19) for this circumstance. The probability that this individual will obtain a correct answer to multiple-choice item g, $P_g'(v_5')$, is presented in column (10). In obtaining these numerical values, (6.1.14) is evaluated for n equal to 6 for each available item in the ability bank. In column (11), one has the expected values of the variance of this individual's ability estimate, $\mathscr{E}_{U_g'}\ \sigma^2(\Theta^* \mid v_5',U_g')$, given the probable outcomes for each available item. Each expected value is obtained by evaluating (6.2.1) where the required inputs are $\sigma^2(\Theta^* \mid v_5')$ or 48.8885 and the row-wise entries in columns (4), (5), (7), (9), and (10). The minimum entry in column (11) is located in order to identify the most informative sixth item for this tailored test. The minimum value of 41.0 identifies Item 8 from the ability bank as Item 6 in this tailored test. Therefore, a 6 is recorded in column 12 and row 8 of Table 8.2.6. Under the selection or rejection analogy, explicit selection or rejection on $\bar{Z}_{6(8)}(v_5')$ subsequent to explicit selections on $\bar{Z}_{1(10)}(v_0')$ and $\bar{Z}_{2(13)}(v_1')$ and explicit rejections on $\bar{Z}_{3(14)}(v_2')$, $\bar{Z}_{4(12)}(v_3')$, and $\bar{Z}_{5(11)}(v_4')$ is expected to restrict most the variance of ability for the selected or rejected subpopulation if *explicit selection* on $\bar{Z}_{6(8)}(v_5')$ actually occurred in this sequence.

This individual then responds with a correct answer to Item 6. Thus the outcome vector v_6' contains the entries: $u_{1(10)}' = 1$, $u_{2(13)}' = 1$, $u_{3(14)}' = 0$, $u_{4(12)}' = 0$, $u_{5(11)}' = 0$, and $u_{6(8)}' = 1$ where the tenth, thirteenth, fourteenth, twelfth, eleventh, and eighth items in the ability bank have been aliased by the subscripts 1, 2, 3, 4, 5, and 6 within this tailored test. Because this individual's answer to Item 6 was correct, the estimate of ability can be revised by evaluating (6.2.2) where the required inputs are $\mu(\Theta^* \mid v_5')$ or 97.6917, $\sigma(\Theta^* \mid v_5')$ or 6.9920, and the unrounded values corresponding to the entries in Table 8.2.6 from columns (4), (5), (7), and (10) for g equal to 8, the subscript of the sixth aliased item. In this case, the evaluation of (6.2.2) yields 99.1180 as the updated estimate of this individual's ability, $\mu(\Theta^* \mid v_6')$. Accordingly, the variance of this estimate of ability can be revised through the use of (6.2.3) where the required inputs are $\sigma^2(\Theta^* \mid v_5')$ or 48.8885, and the unrounded values corresponding to the entries in Table 8.2.6 from columns (4), (5), (6), (7), and (10) for g equal to 8, the subscript of the sixth aliased item. In this case the use of (6.2.3) yields 42.5762 as the updated variance of the estimate of ability, $\sigma^2(\Theta^* \mid v_6')$. The error of this estimate, $\sigma(\Theta^* \mid v_6')$, is 6.5250. Since 6.5250 is less than 6.6667 with respect to the comparison given in (6.2.6), which now satisfies the condition stipulated in (6.2.6), the test is concluded. The item subscript at the termination of testing for this individual, q_i, is, thus, 6.

8.3. Summary of Test Results

In Table 8.3.1, a summary is presented for the test that was tailored to this individual. Column (1) provides the order in which items were chosen and presented within the tailored test. This order of presentation was indexed by the subscript n. In column (2), the subscripts $g^{(n)}$ are presented for the items chosen from the ability bank to form this tailored test. The individual's binary scores on the chosen items, $u_{n[g^{(n)}]}'$, are recorded in column (3). In column (4), the $\mu(\Theta^* \mid v_n')$, the estimates of this individual's ability are given. The variances of these estimates, the $\sigma^2(\Theta^* \mid v_n')$, and the errors of these estimates, the $\sigma(\Theta^* \mid v_n')$, are presented in columns (5) and (6), respectively. In the row where n equals 0, the initial estimate of the individual's ability, $\mu(\Theta^* \mid v_0')$ or 100.0000, the variance of this estimate, $\sigma^2(\Theta^* \mid v_0')$ or 444.4444, and the error of this estimate, $\sigma(\Theta^* \mid v_0')$ or 21.0819, are recorded. In the row where n equals 1, and reading from left to right, the following are recorded: the ability bank subscript for Item 1, 10; this individual's binary score on Item 1 or Item 10 from the ability bank, $u_{1(10)}'$ or 1; the estimate of this individual's ability given the binary score, $\mu(\Theta^* \mid v_1')$ or 111.7567; the variance of this ability estimate given this binary score, $\sigma^2(\Theta^* \mid v_1')$ or 282.4501; and the error of this ability

Table 8.3.1
A Summary of the Tailored Test for Individual i

(1) Item (n)	(2) $g^{(n)}$	(3) $u'_{n|g^{(n)}|}$	(4) $\mu(\Theta^* \mid v'_n)$	(5) $\sigma^2(\Theta^* \mid v'_n)$	(6) $\sigma(\Theta^* \mid v'_n)$
0			100.0000	444.4444	21.0819
1	10	1	111.7567	282.4501	16.8063
2	13	1	120.4269	200.7159	14.1674
3	14	0	108.6462	107.6038	10.3732
4	12	0	101.3935	64.2195	8.0137
5	11	0	97.6917	48.8885	6.9920
6	8	1	99.1180	42.5762	6.5250

Note. $q_i = 6$, $\mu_i(\Theta^* \mid v'_{q_i}) = 99.1180$, $\sigma_i^2(\Theta^* \mid v'_{q_i}) = 42.5762 = (.0958)\ \sigma^2(\Theta^* \mid v'_0)$,
$\epsilon^2 = (6.6667)^2 = 44.4448 = (.1000)\ \sigma^2(\Theta^* \mid v'_0)$.

estimate given the binary score, $\sigma(\Theta^* \mid v'_1)$ or 16.8063. Notice in this tabular presentation that the outcome vector, v'_{q_i}, is summarized in a sequential or row-wise fashion in columns (1), (2), and (3). For instance, the second entry in this vector was $u'_{2(13)} = 1$ which is identically the information recorded in columns (1), (2), and (3) in the row where n equals 2. The other entries in this row are: the individual's ability estimate after two items $\mu(\Theta^* \mid v'_2)$ or 120.4269; the variance of this estimate $\sigma^2(\Theta^* \mid v'_2)$ or 200.7159; and the error of this estimate $\sigma(\Theta^* \mid v'_2)$ or 14.1674. Accordingly, the third entry in the outcome vector, v'_{q_i}, was $u'_{3(14)} = 0$ which is identically the information contained in columns (1), (2), and (3) in the row where n equals 3. The other entries in this row are: $\mu(\Theta^* \mid v'_3)$ or 108.6462, the individual's ability estimate after three items; $\sigma^2(\Theta^* \mid v'_3)$ or 107.6038, the variance of this ability estimate; and $\sigma(\Theta^* \mid v'_3)$ or 10.3732, the error of this ability estimate. Analogous interpretations apply with respect to the remaining rows in Table 8.3.1. However, the row where n corresponds to q_i—6 for this individual—has added significance. The entry in this row and in column (4) or 99.1180 is this individual's "official" estimate of ability since its associated error or 6.5250 was less than ϵ or 6.6667. The value of ϵ is the specified level of precision. In column (5) of this row, a within-individual variance of 42.5762 is reported. This numerical value corresponds to (.0958) $\sigma^2(\Theta^* \mid v'_0)$ which closely approximates (.1000) $\sigma^2(\Theta^* \mid v'_0)$, or ϵ^2, 44.4448. The squared terminal error ϵ^2 sets the lower bound to the index of reliability for the results obtained through the tailored testing process. The lower bound to the index of reliability as well as its implications for practice was discussed earlier in Chapter 7.

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